# STABILITY ANALYSIS OF ORTHOTROPIC PLATES BY A MIXED COMPLEMENTARY ENERGY METHOD

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## SUMMARY

In this paper elastic buckling loads for thin rectangular orthotropic plates are determined using a recently introduced technique [1] which utilizes the principle of a mixed complementary energy approach. Buckling loads are found for various edge and loading conditions of plate. The edge conditions may vary from simply supported all around to clamped plates, with the loading being of uniaxial, biaxial or of shear type. Results using the potential energy method which yield an upper bound to the solution are used for comparison. Convergence of the solution is discussed for different edge and loading conditions of rectangular orthotropic plates.

## FORMULATION

Complementary strain energy, $U^*$, for orthotropic plates may be expressed as

$$U^* = \frac{1}{2} \iint_A \Big[ \frac{1}{(D_{11}D_{22} - D_{12}^2)} (D_{22}M_x^2 - 2D_{12}M_xM_y + D_{11}M_y^2) + \frac{1}{D_{66}} M_{xy}^2 \Big] dxdy \qquad (1)$$

One may eliminate $M_{xy}$ by using equilibrium which yields for the twisting moment

$$M_{xy} = \tfrac{1}{2}\{\int(N_x w_{,x} + N_{xy} w_{,y})dy + \int(N_y w_{,y} + N_{xy} w_{,x})dx$$

$$- \int M_{x,x} dy - \int M_{y,y} dx\} \qquad (2)$$

Also, work done by inplane forces, $W_F$, is given by the following expression

$$W_F = \tfrac{1}{2} \iint_A [N_x (\frac{\partial w}{\partial x})^2 + N_y (\frac{\partial w}{\partial y})^2 + 2N_{xy} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}] \, dxdy \qquad (3)$$

**The principle of mixed complementary energy states that the variation of the quantity ($U^* - W_F$) should be zero, i.e.**

$$\delta(U^* - W_F) = 0 \qquad (4)$$

The same procedure of analysis suggested in [1] is used here. One assumes the following expansions for $w$, $M_x$, and $M_y$

$$w(x,y) = \sum_{i=1}^{N_w} \sum_{j=1}^{N_w} a_{ij} \, \phi_i(x) \, \psi_j(y) \qquad (5)$$

$$M_x(x,y) = \sum_{k}^{N_{M_x}} \sum_{\ell}^{N_{M_x}} b_{k\ell} \, \phi_{k,xx}(x) \, \psi_\ell(y) \qquad (6)$$

$$M_y(x,y) = \sum_{m}^{N_{M_y}} \sum_{n}^{N_{M_y}} C_{mn} \, \phi_m(x) \, \psi_{n,yy}(y) \qquad (7)$$

where

$w(x,y)$ = lateral deflection of the plate

$\phi_i(x), \psi_j(y)$ = eigenfunctions of free vibration of beams having the same boundary conditions as the plate in the x and y directions, respectively (as tabulated in [5])

$N_W$ = number of terms used in the expansion of $w(x,y)$

$M_x, M_y$ = bending moment per unit length of sections of plate perpendicular to the x and y axes, respectively.

$N_{M_x}$, $N_{M_y}$ = number of terms used in the expansion of $M_x$, $M_y$ series, respectively.

In order to formulate the problem, equations (2), (5), (6) and (7) are substituted into equations (1) and (3) and on substituting the results in equation (4) subsequently yields

$$\frac{\partial(U^* - W_F)}{\partial a_{ij}} = \frac{\partial \pi^*}{\partial a_{ij}} = 0 \qquad i,j = 1,2,\ldots,N_W \tag{8}$$

$$\frac{\partial \pi^*}{\partial b_{k\ell}} = 0 \qquad k,\ell = 1,2,\ldots,N_{M_x} \tag{9}$$

$$\frac{\partial \pi^*}{\partial C_{mn}} = 0 \qquad m,n = 1,2,\ldots,N_{M_y} \tag{10}$$

The inplane loads are expressed as

$$\begin{aligned} N_x &= P n_{xx} \\ N_y &= P n_{yy} \\ N_{xy} &= P n_{xy} \end{aligned} \tag{11}$$

where

$P$ = common load parameter

$n_{xx}$, $n_{yy}$, $n_{xy}$ = ratios of inplane forces

Equations (8) to (10) will yield simultaneous algebraic equations of the form :

$$\left( \begin{bmatrix} P^2 C_{11} & P C_{12} \\ P C_{21} & C_{22} \end{bmatrix} - \begin{bmatrix} P B_{11} & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \{0\} \tag{12}$$

where

$[C_{11}]$, $[C_{22}]$, $[B_{11}]$, $\{C_{12}\}$, $\{C_{21}\}$ are submatrices

$$\{q_1\} = \{a_{11} \; a_{12} \; \ldots \; a_{N_W N_W}\} \tag{13}$$

$$\{q_2\} = \{b_{11} \; b_{12} \; \ldots \; b_{N_{M_x} N_{M_x}} \; C_{11} \; C_{12} \ldots C_{N_{M_y} N_{M_y}}\} \tag{14}$$

Eliminating $\{q_2\}$ from equation (12), one gets

$$P^2([C_{11}] - [C_{12}][C_{22}]^{-1} [C_{21}]) \{q_1\} = P[B_{11}]$$

## NUMERICAL RESULTS

A computer program was developed to formulate and solve the problems of the buckling of isotropic and orthotropic thin rectangular plates.

The following cases of plate boundary conditions were studied :

1. All edges simply supported.
2. Simply supported in x-direction and clamped - simply supported in y-direction .
3. Clamped - simply supported on opposite edges.
4. Clamped on all edges .

The following loadings were applied to the plates :

1. $N_x = 1.0$, $N_y = N_{xy} = 0.0$ (Uniaxial buckling)
2. $N_x = N_y = 1.0$, $N_{xy} = 0.0$ (Biaxial buckling)
3. $N_{xy} = 1.0$, $N_x = N_y = 0.0$ (Shear buckling)

Results are presented in Tables I, II and III for a plate clamped on all four edges and subjected to uniaxial, biaxial and shear inplane forces, respectively.

All computations for the eigenvalues were made using material rigidities [2]

$D_{11}$ = 181.8 N.m

$D_{12}$ = 2.897 N.m

$D_{22}$ = 10.34 N.m

$D_{66}$ = 7.17 N.m

## DISCUSSION OF RESULTS AND CONCLUSIONS

1. Regarding the method of solution itself, this method proves to be reliable and versatile. This is because the variables in the energy functional namely, $M_x$, $M_y$ and w, can be modified to satisfy any force and boundary condition and serve as a good approximation to the exact solution.

   Expanding w, $M_x$, $M_y$ in generalized Fourier series gives a greater freedom in controlling the variables that affect the convergence of the solution in comparison to the potential energy method, where only w is expanded in Fourier series. In addition, in the mixed complementary energy method, the number of terms in series expansion for w, $M_x$, and $M_y$ are taken independent of each other, whereas the pure complementary energy approach of Oran [3,4] uses the same number of terms for all variables in the functional.

2. The general program used in the solution was first tested for isotropic plates. This program gave the same numerical values for $P_{cr}$ as those given in [1]. Upon studying the results in more detail and running several cases, it was concluded that it is not necessary for the buckling load to be associated with the fundamental mode shape (i.e. m = 1, n = 1) as was emphasized in [1]. As a matter of fact, it is very difficult to predict the mode shape that is associated with the lowest buckling load. In the results tabulated in [1], the differences between the converged potential energy and mixed complementary energy answers for critical load reach more than 50%,which is difficult to justify. Mixed complementary energy method solutions should converge to answers which should not differ too much from answers given by potential energy method.

3. Based on the above conclusions, several cases of plate buckling problems were run with various loading and boundary conditions. In each of these cases, one observed all values of $P_{cr}$ obtained from the mixed complementary energy method and which are less than the critical buckling load obtained by the upper bound potential energy method. From all such values one chooses the highest value as the critical buckling load using the mixed complementary energy method.

4. One should be careful in choosing the variables $N_w$, $N_{M_x}$, and $N_{M_y}$. For some cases of plate buckling, there will be convergence if $N_w$ is increased together with $N_{M_x}$ and $N_{M_y}$. In other cases, convergence will be observed if $N_{M_x}$ and $N_{M_y}$ are increased at constant $N_w$. The conclusions given in [1] about this point contradicts the conclusion here. In [1], it was suggested not to take $N_w > N_{M_x}$ nor $N_w > N_{M_y}$ since it was claimed that this would result in erroneous eigenvalues for higher modes that could not be simulated by values of $N_{M_x}$ and $N_{M_y}$ lower than $N_w$.

The reasoning given in [1] for the above conclusion was also based on the premise that the correct eigenvalue is the one associated with the fundamental buckling mode. This hypothesis is not correct as one may note results for buckling of simply supported rectangular plates with aspect ratio $a/b > 1$.

5. Buckling of plates loaded by inplane shear was not discussed in [1] at all. There is no predominant form in the eigenvectors in this case, i.e., buckling occurs in a mixed mode. This result makes the hypothesis used by Sundararajan for interpreting buckling loads somewhat ambiguous.

6. In studying the two available complementary energy approaches [1,4], it appears that the method presented in [1] is more versatile in terms of treatment of plates with a variety of boundary conditions. Oran's approach in [4] essentially is based on a Rayleigh-Ritz solution technique which involves approximate expansions for the bending moments, twisting moment and a transverse shear resultant stress function. Such expansions are relatively straightforward for simply supported plates, the only type of boundary conditions for rectangular plates as studied by Oran. Such expansions will, in general, not be so straightforward for plates with other types of boundary conditions, whereas the mixed energy approach of [1] lends itself very conveniently inasmuch as the displacement function w is retained in the final formulation of the problem, alongwith the bending moments $M_x$ and $M_y$.

7. The stability criterion

$$\delta^2 \pi^* = 0$$

may be used for extraction of the eigenvalues in the mixed complementary energy approach. Such a criterion, if implemented successfully, would avoid the multiplicity of eigenvalues yielding stable equilibrium configurations.

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Table I - Uniaxial Buckling of Clamped Plate

| $N_w$ | $N_{M_x}$ | $N_{M_y}$ | Potential Energy | Mixed Complementary Energy |
|---|---|---|---|---|
| | | $N_x = 1.0,\ N_y = N_{xy} = 0.0$ | | |
| 2 | 2 | 3 | 8241.83 | 5285.24 |
| | 3 | 3 | | 3629.85 |
| | 4 | 4 | | 4939.59 |
| | 5 | 5 | | 4939.58 |
| | 6 | 6 | | 4962.53 |
| | 7 | 7 | | 4962.52 |
| 3 | 3 | 3 | 8001.84 | 4405.20 |
| | 4 | 4 | | 4939.59 |
| | 5 | 5 | | 6174.57 |
| | 6 | 6 | | 6175.75 |
| | 7 | 7 | | 6175.75 |
| 4 | 4 | 4 | 8001.85 | 7914.74 |
| | 5 | 5 | | 7914.73 |
| | 6 | 6 | | 7916.09 |
| | 7 | 7 | | 7916.07 |
| 5 | 5 | 5 | 7988.04 | 7914.73 |
| | 6 | 6 | | 7916.07 |
| | 7 | 7 | | 7916.04 |
| 6 | 6 | 6 | 7986.41 | 7732.66 |
| | 7 | 7 | | 7732.65 |

Table II - Biaxial Buckling of Clamped Plate

| | | $N_x = N_y = 1.0,\ N_{xy} = 0.0$ | | |
|---|---|---|---|---|
| $N_w$ | $N_{M_x}$ | $N_{M_y}$ | Potential Energy | Mixed Complementary Energy |
| 2 | 2 | 3 | 2568.21 | 1344.06 |
| | 3 | 3 | | 2444.97 |
| | 4 | 4 | | 2444.98 |
| | 5 | 5 | | 2075.57 |
| | 6 | 6 | | 2088.22 |
| | 7 | 7 | | 2088.22 |
| 3 | 3 | 3 | 2466.82 | 2115.31 |
| | 4 | 4 | | 2115.32 |
| | 5 | 5 | | 1997.18 |
| | 6 | 6 | | 2009.35 |
| | 7 | 7 | | 2009.34 |
| 4 | 4 | 4 | 2416.13 | 2332.86 |
| | 5 | 5 | | 2335.31 |
| | 6 | 6 | | 2337.25 |
| | 7 | 7 | | 2337.25 |
| 5 | 5 | 5 | 2416.03 | 2335.08 |
| | 6 | 6 | | 2336.99 |
| | 7 | 7 | | 2337.00 |
| 6 | 6 | 6 | 2415.76 | 2335.47 |
| | 7 | 7 | | 2335.47 |

Table III - Shear Buckling of Clamped Plate

| | | $N_x = N_y = 0.0, \quad N_{xy} = 1.0$ | | |
|---|---|---|---|---|
| $N_w$ | $N_M$ | $N_{M_y}$ | Potential Energy | Mixed Complementary Energy |
| 2 | 2 | 2 | 12781.14 | 198.97 |
| 3 | 3 | 3 | 8114.29 | 2090.00 |
| | 4 | 4 | | 4140.09 |
| | 5 | 5 | | 4601.69 |
| | 6 | 6 | | 4625.14 |
| | 7 | 7 | | 4625.14 |
| | 8 | 8 | | 4625.14 |
| 4 | [illegible] | 1 | 7309.82 | 1297.75 |
| | [illegible] | 2 | | 1576.02 |
| | [illegible] | 3 | | 3476.22 |
| | [illegible] | 4 | | 7195.23 |
| 5 | [illegible] | 5 | 7269.67 | 5055.68 |
| | [illegible] | 6 | | 5112.13 |
| | [illegible] | 7 | | 5112.13 |
| 6 | [illegible] | 1 | 7268.71 | 5539.89 |
| | [illegible] | 2 | | 5539.99 |
| | 3 | 3 | | 5540.11 |
| | 4 | 4 | | 5540.21 |
| | 5 | 5 | | 5541.76 |
| | 6 | 6 | | 5365.74 |
| | 7 | 7 | | 5365.77 |
| 7 | 7 | 7 | 7108.10 | 5366.72 |
| | 8 | 8 | | 5366.72 |