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Analytical and Numerical Treatment of Maxwell's Equations

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DEDICATION

First of all thanks to God

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and also my family in law.

الاقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان :

Analytical and Numerical Treatment of Maxwell's Equations

أقر بأن ما اشتملت عليه هذه الرسالة انما هي نتاج جهدي الخاص، باستثناء ما تمت الاشارة اليه حيثما ورد، و أن هذه الرسالة ككل، أو أي جزء منها لم يقدم لنيل أية درجة أو لقب علمي أو بحثي لدى أية مؤسسة تعليمية أو بحثية أخرى .

Declaration

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

Student's Name : **اسم الطالب :**

Signature : **التوقيع:**

Date : **التاريخ :**

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Analytical and Numerical Treatment of Maxwell's Equations

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Abstract

Maxwell's equations are one of the most important models in different fields. It describes electromagnetic phenomena such as micro, radios and radar waves.

The modeling of systems involving electromagnetic waves is widely spread and has attracted the attention of many authors and researchers. In this work, we will present some important analytical and numerical aspects of Maxwell's equations. We will review some basic properties of electromagnetic theory, namely: electromagnetic fields, magneto-static fields, and time varying fields. Moreover, we will use these physical properties to derive Maxwell's equations in various forms. Though, it is well known that Maxwell's equations are hard to solve analytically, however, we will attempt to use some well known analytical methods to solve these equations in some particular domains such as a sphere and a circular cylinder. Such analytical methods include: separation of variables, series expansion method, conformal mapping and integral methods such as Laplace transforms and cosine and sine Fourier transforms.

Numerical methods for solving Maxwell's equations are extensively used nowadays and are usually referred to as Computational Electro-magnetic

(CEM). Here the Finite Difference and Finite Difference Time Domain Method (FDTDM) known for its simplicity and efficiency will be proposed to solve Maxwell's equations. And the Yee Algorithm will also be illustrated. Moreover, the convergence, stability and error analysis for these numerical methods will also be investigated.

Chapter One
Preliminaries

1.1 Introduction

Electromagnetic phenomena play a very prominent role in the modern age. The number of electric machines are uses on a daily basis without thinking about it is large, and only becomes imminent during an unfortunate power failure. Physicists have succeeded quite well in formulating the laws to which these phenomena must adhere. In the late 1860s, J.C. Maxwell constructed the mathematical framework combining the phenomenological finding of his predecessors concerning electromagnetism. This can be viewed as the birth of mathematical physics. Maxwell's achievement has stimulated many other people since, to construct similar basic sets of equations to describe other fields in physics, but it turned that not every field of physics could be as nicely and elegantly described as electromagnetism. The set of basic equations Maxwell constructed became known as the Maxwell Equations and are given in their differential formulation by [5,19]:

$$\frac{\partial}{\partial t}\mathbf{H}(t) = -\frac{1}{\mu}\nabla \times \mathbf{E}(t).$$

$$\frac{\partial}{\partial t}\mathbf{E}(t) = \frac{1}{\varepsilon}\left(\nabla \times \mathbf{H}(t) - \mathbf{J}(t)\right).$$

$$\operatorname{div} \varepsilon\mathbf{E}(t) = \rho(t).$$

$$\operatorname{div} \mu\mathbf{H}(t) = 0.$$

Here

- $\mathbf{E}$ is the electric field intensity (in volts/meter).
- $\mathbf{H}$ is the magnetic field intensity (in amperes/meter).
- $\mathbf{J}$ is the total electric current density (in amperes/meter^2), and equates $\mathbf{J} = \mathbf{J}_{source} + \sigma \mathbf{E}$, where $\mathbf{J}_{source}$ satisfies the continuity equation
$$\nabla \mathbf{J}_{source} = -\left(\frac{\partial}{\partial t}\right)\rho .$$
- μ is the magnetic permeability (in henrey/meter).

In this work we assume that the magnetic permeability does not depend on time (or consequently frequency).

- ε is the electric permittivity (in farads/meter), in vacuum we have $c = (\mu_0 \varepsilon_0)^{-1/2}$. The electric permittivity is also assumed not to depend on time.
- σ is the electric conductivity (in siemens/meter),
- ρ is the charge density (in colomb/meter^3).

The variety of applications based on electromagnetism is enormous. Unfortunately, in the case where a solution of the Maxwell equations is required, it is not possible to solve them analytically. It can be solved analytically only for a simple domains such as a sphere and an infinite circular cylinder [31]. Numerical methods for the Maxwell's equations are usually referred to as Computational Electromagnetic (CEM).

The modeling of systems are involving electromagnetic waves is now widely done through the solution of the time domain Maxwell's equations

on space grid. Such systems were solved with many different methods, the first method for the numerical simulation of time dependent electromagnetic wave, the Finite Difference Time Domain Method (FDTDM), was proposed by K. Yee [31], it is clear that FDTDM, known due its simplicity and efficiency, but it is difficult to generalize to unstructured non-cartesian grids and suffer from the inaccurate representation of the solution on curred boundaries [31]. Moreover, FDTDM has accuracy limitations, for second order accuracy severely limits their ability to correctly represent wave motion over long distances unless the grid is prohibitively fine [35]. Many different methods have been proposed which are based on unstructured grids and can deal with complex geometries. Like Finite Element Time Domain Method (FETDM) [7,21]. There are two difficulties appear when using the standard Finite Element Method (FEM). First, the method generally used on a globally conforming mesh, that mean, a mesh without hanging (connecting) nodes or mismatch of mesh points along internal boundaries. Second, how the corner singularities can be represented [6,22]. The Discontinuous Galerkin Methods (DGM) [6,14,17,20,33] are based on discontinuous finite element spaces. It is easily handle elements of various types and shapes, irregular non-conforming meshes and even locally varying polynomial degree. Moreover, continuity is weakly enforced across mesh interfaces by adding bilinear forms [33]. Either on tetrahedral meshes using Lagrange polynomials [10,32] or on hexahedral meshes using products of Lagrange polynomials [10,32]. On the other hand Variational Iteration

Method(VIM), [1,2,8,11,16,29,30] is proposed by J. He [16,30] on a modification of a general Lagrange multiplier method. This technique provides a sequence of functions which converges to the exact solution of the problem. It has been shown that this procedure is a powerful tool for solving various kinds of problems. This technique solves the problem without any need to discretization of the variables, therefore it is not effected by computation round off errors and one is not forced with necessity of large computer memory and time. This thesis is organized as follows: In chapter one, we review some of electromagnetic field theory. This includes electromagnetic fields, magnetostatic fields and time-varying fields. In chapter two, derivation of Maxwell's equations is presented in different forms. Chapter three, deals with some analytical methods used to solve electromagnetic problems. The most commonly used analytical methods that are presented here are: the separation of variables, eigen function expansion method, conformal mapping and integral methods. In chapter four we present some numerical methods for solving Maxwell's equations. These methods are the Finite Difference Method and Finite Difference Time Domain Method, and we also present Accuracy and Stability of FD Solutions , Convergence, Consistency, and Stability analysis of FD methods and the local truncation error.

1.2 Preliminaries in Physics and Mathematics

Our modern society relies on electromagnetic devices and systems such as: radio, internet, microwave ovens, satellite communication systems, medical imaging systems and many more [26]. The understanding of electromagnetic phenomena is treated by electromagnetic field theory the study of interactions between electric charges at rest and in motion. (Electric charges in motion are often referred to as electric currents.). Electromagnetic field theory describes the interactions between electric charges by Maxwell's equations, a system of coupled partial differential equations that relate sources (charges and currents) to the electromagnetic fields and fluxes. To this end, we will state some important theorems together with definitions for some physical quantities that will help us later to derive the Maxwell's equations.

Electromagnetic Feld Theory

I must begin with answering some questions which relate directly to electromagnetism [15], these questions are: What is a field? Is it scalar field or vector field? What is the nature of the field? The primary purpose of this text is to answer some of these questions pertaining to electromagnetic fields.

Field Concept

The region throughout which a physical quantity is so specified is a field, a field may be scalar or vector. The scalar field is specified by a single number at each point. Well-known examples are temperature and pressure of a gas [15]. A vector field is specified by both a magnitude and a direction at each point in space,well–known examples are velocity and acceleration.

Static Field

If a field does not vary with time we refer to it as *a static field*. Static fields are also known as *time-invariant fields* [15,26]. The fields produced by stationary charges are called (electrostatics), and the fields created by a steady motion of charges are called (magnetostatics).

Time Varying Field

This field deals with charges in motion which create a current [15,26]. If the movement of charges is restricted in such a way that the resulting current is constant in time, the field thus created is called magnetic field. Since the current is constant in time, the magnetic field is also constant in time.

Vector Calculus

Vector calculus (or vector analysis) is a branch of mathematics concerned with differentiation and integration of vector fields, primarily in 3

dimensional Euclidean space $\mathbf{R}^3$.The term "vector calculus" is sometimes used as a synonym for the broader subject of multivariable calculus [15,34], which includes vector calculus as well as partial differentiation and multiple integration. Vector calculus plays an important role in differential geometry and in the study of partial differential equations. It is used extensively in physics and engineering, especially in the description of electromagnetic fields, gravitational fields and fluid flow.

The Gradient of a Scalar Field

In vector calculus, the gradient of a scalar field is a vector field that points in the direction of the greatest rate of increase of the scalar field [15,34], and whose magnitude is the greatest rate of change. Which is often written using the standard vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$:

$$\nabla \times \mathbf{F} = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k}. \tag{1.1}$$

Divergence of Vector Field

In vector calculus, divergence is a vector operator that measures the magnitude of a vector field's source or sink at a given point, in terms of a signed scalar. More technically, the divergence represents the volume density of the outward flux of a vector field from an infinitesimal volume around a given point. For example, consider air as it is heated or cooled. The relevant vector field for this example is the velocity of the moving air

at a point, the divergence of a continuously differentiable vector field is equal to the scalar-valued function:

$$div\ \mathbf{F} = \nabla \bullet \mathbf{F} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}. \tag{1.2}$$

The Divergence Theorem

In vector calculus, the divergence theorem, also known as *Gauss' theorem* is a result that relates the flow (that is, flux) of a vector field through a surface to the behavior of the vector field inside the surface [15,26,34]. More precisely, the divergence theorem states that the outward flux of a vector field through a closed surface is equal to the volume integral of the divergence of the region inside the surface. And expressed mathematically as:

$$\iiint_V (\nabla \bullet \mathbf{F}) dV = \oiint_S (\mathbf{F} \bullet \mathrm{n}) dS. \tag{1.3}$$

This equation relates the volume integral of the divergence of a vector field to the surface integral of its normal component.

The Curl of Vector Field

The line integral of a vector field $\vec{\mathbf{F}}$ around a closed path is called the circulation of $\vec{\mathbf{F}}$ and the curl of $\vec{\mathbf{F}}$ is its measure. The direction of the curl is the axis of rotation, as determined by the right-hand rule, and the magnitude of the curl is the magnitude of rotation [15,26]. A vector field whose curl is zero is called irrotational or conservative. The corresponding

form of the fundamental theorem of calculus is Stokes' theorem, which relates the surface integral of the curl of a vector field to the line integral of the vector field around the boundary curve. Expanded in Cartesian coordinates, $\nabla \times \mathbf{F}$ is, for $\mathbf{F}$ composed of [F_x, F_y, F_z]:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}.$$

Where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the unit vectors for the *x*, *y*, and *z* axes, respectively. This expands as follows:

$$\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}\,. \qquad (1.4)$$

Thus, we will always write curl $\mathbf{F}$ as $\nabla \times \mathbf{F}$.

Stokes' Theorem

It states that the integral of the normal component of the curl of a vector field over an area is equal to the line integral of the vector field along the curve bounding the area [26]. The only contribution is from the integration over the path *C*. thus:

$$\int_S (\nabla \times \mathbf{F}).\boldsymbol{ds} = \oint_C \mathbf{F}.\boldsymbol{dl}\,. \qquad (1.5)$$

1.3 Classification of Fields

In our study of electromagnetic fields, we will find that fields have four basic classifications. In solving field problems it is necessary to know which class of field we are working with because this will dictate the procedure we must use to solve the problem [15]. Therefore let us now examine the features of the fields belonging to each class.

Class 1 field : A vector field $\mathbf{F}$ is to be of class 1 everywhere in a given region if $\nabla \bullet \mathbf{F} = 0$ and $\nabla \times \mathbf{F} = 0$ [15]. However, if the curl of a vector is zero, then the vector can be written in terms of a gradient of a scalar function f , that is $\mathbf{F} = -\nabla f$. Electrostatic fields in charge free medium and magnetic fields in current free medium are examples of class 1 field.

Class 2 field : A vector field $\mathbf{F}$ is to be of class 2 everywhere in a given region if $\nabla \bullet \mathbf{F} \neq 0$ and $\nabla \times \mathbf{F} = 0$ [15], once again $\nabla \times \mathbf{F} = 0$ implies $\mathbf{F} = -\nabla f$, because $\nabla \bullet \mathbf{F} \neq 0$, we can write it as $\nabla \bullet \mathbf{F} = \rho$, where ρ is either a constant or a known function within the region, thus : $\nabla^2 f = -\rho$ which is Poisson's equation, thus class 2 field can be found by solving Poisson's equation within the constraints of the boundary condition. We can then find the vector field $\mathbf{F}$ as $\mathbf{F} = -\nabla f$. An electrostatic field in a charged region is an example of class 2 field.

Class 3 field: A vector field $\mathbf{F}$ is to be of class 3 everywhere in a given region if $\nabla \bullet \mathbf{F} = 0$ and $\nabla \times \mathbf{F} \neq 0$, if the divergence of a vector is zero [15], then the vector can be expressed in terms of the curl of another vector. For

$\nabla \bullet \mathbf{F} = 0$, we can express $\mathbf{F}$ as $\mathbf{F} = \nabla \times \mathbf{A}$ .Where $\mathbf{A}$ is another vector field, because $\nabla \times \mathbf{F} \neq 0$, we can write it as $\nabla \times \mathbf{F} = \mathbf{J}$. Where $\mathbf{J}$ is a known vector field. Substituting $\mathbf{F} = \nabla \times \mathbf{A}$, we get $\nabla \times \nabla \times \mathbf{A} = \mathbf{J}$, using the vector field identity, we can express this equation as $\nabla(\nabla \times \mathbf{A}) - \nabla^2 \mathbf{A} = \mathbf{J}$. For $\mathbf{A}$ to be a unique vector field, we must define its divergence. If we set an arbitrary constraint that $\nabla \bullet \mathbf{A} = 0$, thus we obtain, $\nabla^2 \mathbf{A} = -\mathbf{J}$. Which is called Poisson's vector equation, therefore class 3 fields require a solution of Poisson's vector equation. The vector field $\mathbf{F}$ can be computed from $\mathbf{A}$ as $\mathbf{F} = \nabla \times \mathbf{A}$. The constraint $\nabla \bullet \mathbf{A} = 0$ is known as *Coulomb's gauge*. The magnetic field within a current-carrying conductor falls into class 3 field.

Class 4 field : for a vector field $\mathbf{F}$ to be of class 4, neither its divergence nor its curl is zero [15]. Then we can decompose $\mathbf{F}$ into two vector fields $\mathbf{G}$ and $\mathbf{H}$ such that $\mathbf{G}$ satisfies class 3 and $\mathbf{H}$ satisfies class 2 requirements. That is, $\mathbf{F} = \mathbf{G} + \mathbf{H}$, $\nabla \bullet \mathbf{G} = 0$, $\nabla \times \mathbf{G} \neq 0$, $\nabla \times \mathbf{H} = 0$, and $\nabla \bullet \mathbf{H} \neq 0$. Thus $\mathbf{H} = -\nabla f$ and $\mathbf{G} = \nabla \times \mathbf{A}$, lead us to conclude that $\mathbf{F} = \nabla \times \mathbf{A} - \nabla f$.

Hydrodynamic fields in a compressible medium are examples of class 4 field.

1.4 Charges and Coulomb's Law

Coulomb stated that the force between two very small objects separated in a vacuum or free space by a distance which is large compared to their size is proportional to the charge on each and inversely proportional to the square of the distance between them. Expressed mathematically [25,34],

$$F = k\,\frac{Q_1 Q_2}{R^2}. \tag{1.6}$$

Where Q_1 and Q_2 are the positive or negative quantities of charge , R is the distance, and k is the proportionality constant, k is written as

$$k = \frac{1}{4\pi\varepsilon_0}.$$

The constant ε_0 is called the permittivity of free space and has the magnitude

$$\varepsilon_0 = 8.854\times10^{-12} = \frac{1}{36\pi}10^{-9}$$

Coulomb's law is now

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}. \tag{1.7}$$

Electric Field Intensity

The electric field intensity **(**or electric field strength**)** denoted by **E** is the force per unit charge when placed in the electric field [34]. Thus the electric field intensity **E** simply is given as

$$\mathrm{E} = \frac{\mathrm{F}}{Q} \tag{1.8}$$

Electric Flux and Electric Flux Density

If we place a test charge at a point in an electric field and allow it to move, then the force acting on the test charge will move it along a certain path [25,26]. This path is called a line of force or a flux line. It is customary to

state that the number of lines of force due to a charge is equal to the magnitude of the charge in coulombs. The field lines are said to represent the electric flux. These electric flux lines have no real existence, but they are a useful concept in the representation of the electric field. One can realize that the *electric flux density* **D** can be defined in terms of electric field intensity **E** as,

$$\mathbf{D}=\varepsilon_0 \mathrm{E} \quad . \tag{1.9}$$

Substituting for **E** due to a point charge Q in the equation above, we obtain the electric flux density at a radius ***r*** as,

$$\mathrm{D}=\frac{Q}{4\pi r^2}\mathbf{a}_r \quad . \tag{1.10}$$

The Electric Flux

We can now define the electric flux Ψ in terms of electric flux density **D** as,

$$\Psi = \int d\Psi = \oint_{\substack{closed \\ surface}} \mathrm{D}_s . d\mathbf{S} \quad . \tag{1.11}$$

Where $d\mathbf{S}$ is the differential surface element on surface S [33]. The flux passing through surface S is maximum if **D** and $d\mathbf{S}$ are in the same direction.

$$\mathrm{D}=\int_{\mathrm{vol}} \frac{\rho_v dv}{4\pi R^2}\mathrm{a}_r . \tag{1.12}$$

Gauss's Law

Gauss's law states that the net outward flux passing through a closed surface is equal to the total charge enclosed by that surface [12,34], that is,

$$\oint_S \mathrm{D}_S \,.\, \boldsymbol{dS} = \text{charge enclosed} = Q. \tag{1.13}$$

Gauss's law can also be expressed in terms of electric field intensity in free space as,

$$\oint_S \mathbf{E} \cdot \boldsymbol{ds} = \frac{Q}{\varepsilon_0}. \tag{1.14}$$

If the charges are distributed in a volume bounded by a surface the eq.(1.13) Can be written as

$$\oint_S \mathbf{D}.\boldsymbol{ds} = \int_v \rho_v dv\,. \tag{1.15}$$

By applying the divergence theorem for the eq.(1.15) we now obtain

$$\int_v \nabla.\mathbf{D} dv = \int_v \rho_v dv\,. \tag{1.16}$$

This must be true for any volume v bounded by a surface, so that the two integrals must be equal. Thus

$$div\ \mathbf{D} = \rho_V\,. \tag{1.17}$$

or

$$\nabla.\mathbf{D} = \rho_V\,. \tag{1.18}$$

This equation is called the point or differential form of Gauss's law, which states that: lines of electric flux emanate from any point in space at which there exists a positive charge density. If the charge density is negative, the lines of electric flux converge toward the point. Now Gauss's law may be written in terms of the charge distribution as

$$\oint_s \mathbf{D}_s . d\mathbf{S} = \int_{\text{vol}} \rho_V \, dv \, . \tag{1.19}$$

Chapter Two
Modeling of Maxwell's equations

2. Modeling of Maxwell's equations

2.1 Maxwell's First Equation

Our aim in this chapter is to use the physical quantities and the theorems presented in chapter one to derive the Maxwell's four equations. Now using the divergence theorem as it relates to electric flux density. The divergence of **A** is defined as [12,34],

$$\text{Divergence of } \mathbf{A} = \text{div } \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A}.d\boldsymbol{S}}{\Delta V} \ . \qquad (2.1)$$

Where the divergence of the vector flux density **A** is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero. The expression developed above for the electric flux density may be written as :

$$\text{div } \mathbf{D} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D} \ .d\boldsymbol{S}}{\Delta V} \ .$$

$$\text{div}\mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \ .$$

and

$$\text{div}\mathbf{D} = \rho_v \ . \qquad (2.2)$$

Using the concept of divergence as defined in (2.1) and (2.2) the Gauss's law states

$$\oint_S \mathbf{A}.d\mathbf{S} = Q \ . \qquad (2.3)$$

Per unit volume

$$\frac{\oint_S \mathbf{A}.d\mathbf{S}}{\Delta v} = \frac{Q}{\partial v} \ .$$

And as the volume shrinks to zero

$$\lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \ .d\mathbf{S}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v} \ . \qquad (2.4)$$

We should see div **D** on the left and volume charge density on the right,

$$\text{div } \mathbf{D} = \rho_V$$

or $$\nabla.\mathbf{D} = \rho_v \ . \qquad (2.5)$$

This is the *first of Maxwell's four equations* as they apply to electrostatic and steady magnetic fields, and it states that the electric flux per unit volume leaving a vanishingly small volume unit is exactly equal to the volume charge density there. This equation is called the *point form of Gauss's law* or the differential form of Gauss's law [12], and conversely, Gauss's law is recognized as the integral form of *Maxwell's first equation*.

Starting from Gauss's law,

$$\oint_S \mathbf{D}.\boldsymbol{ds} = Q \ . \qquad (2.6)$$

and letting $$Q = \oint_{vol} \rho_v dv \ . \qquad (2.7)$$

and then replacing ρ_v from eq.(2.5) ,

So we obtain

$$\oint_S \mathbf{D}.\boldsymbol{ds} = Q = \int_{\text{vol}} \rho_v dv = \int_{\text{vol}} \nabla.\mathbf{D}\, dv \tag{2.8}$$

The first and last expressions constitute the divergence theorem,

$$\oint_S \mathbf{D}.\boldsymbol{ds} = \int_{\text{vol}} \nabla.\mathbf{D} dv \tag{2.9}$$

Which may be stated as follows: The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by that closed surface.

2.2 Maxwell's second equation

Definition of Potential Difference and Potential

Another way we will discuss now of obtaining the electric field intensity $\mathbf{E}$ is from the electric scalar potential V to be defined [34]. If we wish to move a point charge Q from point A to point B in an electric field $\mathbf{E}$, then from Coulomb's law the force on Q is $\mathbf{F} = Q\mathbf{E}$ so that the work done in displacing the charge by $d\mathbf{L}$ is

$$d\mathrm{W} = -\mathbf{F}.d\mathbf{L} = -Q\,\mathrm{E}\,d\mathbf{L} \tag{2.10}$$

The negative sign indicates that the work is being done by an external agent. Thus the total work done, or the potential energy required in moving Q from A to B is

$$W = -Q\int_{\mathrm{A}}^{\mathrm{B}} \mathbf{E}.d\mathbf{L}\,. \tag{2.11}$$

Dividing W by Q in the equation above gives the potential energy per unit charge. This quantity denoted by V_{AB} is known as the potential difference between points A and B, thus

$$V_{AB} = \frac{W}{Q} = -\int_A^B \mathbf{E}.d\mathbf{L} \ . \tag{2.12}$$

If the field **E** is due to a point charge Q located at the origin, then

$$\mathrm{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\mathbf{a}_r \ . \tag{2.13}$$

and so

$$V_{AB} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\varepsilon_0 r^2}\mathbf{a}_r.dr \ \mathbf{a}_r$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right].$$

$$V_{AB} = V_B - V_A \ . \tag{2.14}$$

It is customary to choose infinity as reference, that is; we assume the potential at infinity is zero. Thus if $V_A = 0$ as $r_A \to \infty$, then the potential at any point $r_B \to r$ due to a point charge Q located at the origin is

$$V = \frac{Q}{4\pi\varepsilon_0 r} \ . \tag{2.15}$$

The potential difference V_{AB} can be found generally from

$$V_{AB} = V_B - V_A = -\int_A^B \mathbf{E}.d\mathbf{L} = \frac{W}{Q} \ . \tag{2.16}$$

The Relationship Between E and V

As mentioned before, the potential difference between two points A and B is independent of the path taken [12,34], hence,

$$V_{BA} = -V_{AB}$$

That is,

$$V_{BA} + V_{AB} = \oint \mathbf{E}.d\mathbf{L} = 0 . \qquad (2.17)$$

or

$$\oint \mathbf{E}.d\mathbf{L} = 0 . \qquad (2.18)$$

This shows that the line integral of $\mathbf{E}$ along a closed path must be zero. Physically, this implies that no net work is done in moving a charge along a closed path in an electrostatic field. Applying Stokes's theorem to equation (2.18) gives,

$$\oint \mathbf{E}.d\mathbf{L} = \int (\nabla \times \mathbf{E}).d\mathbf{s} = 0 . \qquad (2.19)$$

or simply

$$\nabla \times \mathbf{E} = 0 \qquad (2.20)$$

Any vector field which satisfies eq.(2.20) is said to be conservative field, or *irrotational* [26,34], thus an electrostatic field is a *conservative field*. Eq. (2.20) is referred to as *The second Maxwell's equation* for static electric field .

From the way we defined potential,

$$V = -\int \mathbf{E}.d\mathbf{L} . \tag{2.21}$$

and thus

$$\mathbf{E} = -\nabla V \ . \tag{2.22}$$

That is the electric field intensity is the gradient of *V,* the negative sign shows that the direction of **E** is opposite to the direction in which *V* increases. Since the curl of the gradient of a scalar function is always zero $(\nabla \times \nabla V = 0)$ this implies that **E** must be a gradient of some scalar function.

2.3 Maxwell's Third Equation

Current and Current Density

Electric charges in motion constitute a current [25,34]. So current is defined as a rate of movement of charge passing a given reference point of one coulomb per second, it is symbolized as ***I***,and therefore,

$$I = \frac{dQ}{dt} \ . \tag{2.23}$$

The current density is the electric current per unit area of cross section, and it's a vector represented by **J** .The increment of the current ΔI crossing an incremental surface ΔS normal to the current density is

$$\Delta I = J_N \Delta S \ .$$

and in the case where the current density is not perpendicular to the surface

$$\Delta I = \mathrm{J}.\Delta S \ .$$

The total current is obtained by integrating

$$I = \int_S \mathbf{J}.d\mathbf{s} \ . \tag{2.24}$$

Current density may be related to the velocity of volume charge density at a point in general by

$$\mathbf{J} = \rho_V \upsilon \ . \tag{2.25}$$

This last result shows clearly that the charge in motion constitutes a current, we call this type of current *convention current*, and **J** or is the convention current density.

Continuity of Current

The principle of conservation of charge states simply that charges can be neither created nor destroyed [12,26,34]. The continuity equation follows from this principle when we consider any region bounded by a closed surface. The current through the closed surface is

$$I = \oint_S \mathbf{J}.d\mathbf{s} \ . \tag{2.26}$$

And this outward flow of positive charge must be balanced by a decrease of positive charge (or perhaps an increase of negative charge) within the closed surface. If the charge inside the closed surface is denoted Q_i then the

rate of decrease is $-dQ_i/dt$ and the principle of conservation of charge requires

$$I = \oint_S \mathbf{J}.d\mathbf{s} = -\frac{dQ_i}{dt} \ . \tag{2.27}$$

The minus sign is to indicate the outward-flowing current. This equation is in the integral form of the continuity equation, and the differential or point form is obtained by using the divergence theorem to change the surface integral into a volume integral :

$$\oint_S \mathbf{J}.d\mathbf{s} = \int_{\text{vol}} (\nabla.\mathbf{J})\, dv \, . \tag{2.28}$$

We next represent the enclosed charge Q_i by the volume integral of charge density :

$$\int_{\text{vol}} (\nabla.\mathbf{J}) dv = -\frac{d}{dt} \int_{\text{vol}} \rho_V \ dv \ . \tag{2.29}$$

If we keep the surface constant, the derivative becomes a partial derivative and may appear within the integral,

$$\int_{\text{vol}} (\nabla.\mathbf{J})\ dv = \int_{\text{vol}} -\frac{\partial \rho_v}{\partial t} \ dv \ . \tag{2.30}$$

Since the expression is true for any volume, however small, it is true for an incremental volume :

$$(\nabla.\mathbf{J})\ \Delta v = -\frac{\partial \rho_v}{\partial t} \Delta v \ . \tag{2.31}$$

From which we have our point form of the continuity equation,

$$(\nabla . \mathbf{J}) = -\frac{\partial \rho_v}{\partial t} \ . \tag{2.32}$$

This is the third of Maxwell's equations which indicates that the current, or charge per second diverging from a small volume per unit volume is equal to the time rate of decrease of charge per unit volume at every point.

2.4 Maxwell's fourth equation

Biot-Savart Law

We assume a current ***I*** flowing in a different vector length of the filament ***dl*** , the law of Biot-Savart then states that at any point P the magnitude of the magnetic field intensity which is denoted by **H** produced by the differential element is proportional to the product of the current, the magnitude of the differential length, and the sin of the angle lying between the filament and a line connecting the filament to the point P at which the field is desired, also the magnitude of the magnetic field intensity is inversely proportional to the square of the distance from the differential element to the point P [34]. The Biot-Savart law described above can be written concisely using vector notation as :

$$d\mathrm{H} = \frac{Id\mathrm{L} \times \mathbf{a}_R}{4\pi R^2} = \frac{Id\mathrm{L} \times \mathrm{R}}{4\pi R^3} \ . \tag{2.33}$$

The Biot-Savart law is sometimes called *Ampere's law for current element*. It is impossible to check experimentally the law of Biot-Savart as expressed

in eq. (2.33), because the differential current element cannot be isolated. We have restricted our attention to direct currents only, since the charge density is not a function of time. The continuity equation

$$\mathrm{V}.\mathbf{J} = -\frac{\partial \rho_v}{\partial t} \ .$$

therefore shows that

$$\mathrm{V}.\mathbf{J} = 0 \ . \tag{2.34}$$

or by applying the divergence theorem we obtain

$$\oint_S \mathbf{J}.d\mathbf{s} = 0 \ . \tag{2.35}$$

The total current crossing any closed surface is zero, the condition may be satisfied only by assuming a current flow around a closed path.

$$\mathbf{H} = \oint \frac{Id\mathrm{L} \times \mathbf{a}_R}{4\pi R^2} \ . \tag{2.36}$$

Ampere's Circuital Law

Ampere's circuital law states that: *the line integral of* **H** *about any closed path is exactly equal to the direct current enclosed by that path* [12,34], and expressed mathematically as;

$$\oint \mathbf{H} \ .d\mathbf{L} = I \ . \tag{2.37}$$

The application of Ampere's circuital law involves finding the total current enclosed by a closed path. Our objective here is to obtain the point form of

Ampere's circuital law. Which is also may be written in terms of the vector operator,

$$\text{curl}\,\mathbf{H} = \nabla \times \mathbf{H}\ . \tag{2.38}$$

We may now describe the curl as *circulation per unit area*. The closed path is vanishingly small, and curl is defined at a point. The curl of $\mathbf{E}$ must be zero, for the circulation is zero. The curl of $\mathbf{H}$ is not zero, however the circulation of $\mathbf{H}$ per unit area is the current density by Ampere's circuital law. We may now combine the two eq.(2.37) & (2.38) and have,

$$\text{curl}\,\mathbf{H} = \nabla \times \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z = \mathbf{J}$$

And write the point form of Ampere's circuital law,

$$\nabla \times \mathbf{H} = \mathbf{J}\ . \tag{2.39}$$

This is the third of Maxwell's four equations as they apply to non-time varying conditions. It is easy now to obtain Ampere's Circuital law from $\nabla \times \mathrm{H} = \mathrm{J}$. we merely have to dot each side by $d\mathbf{S}$, integrate each side over the same (open) surface S , and apply stokes' theorem:

$$\int_S (\nabla \times \mathbf{H}).d\mathbf{s} = \int_S \mathbf{J}.d\mathbf{s} = \oint \mathbf{H}.d\mathbf{L}\ . \tag{2.40}$$

The integral of the current density over the surface S is the total current ***I*** passing through the surface, and therefore:

$$\oint \mathbf{H}.d\mathbf{L} = I \ . \tag{2.41}$$

$$\oint \mathbf{H}.d\mathbf{L} = I = \int_S \mathbf{J}.d\mathbf{s} \ . \tag{2.42}$$

Magnetic Flux and Magnetic Flux Density

In free space we define the magnetic flux density **B** as $\mathbf{B} = \mu_0\mathbf{H}$, where μ_0 is the permeability of free space and has the value $\mu_0 = 4\pi \times 10^{7}$ .The relation $\mathbf{B} = \mu_0\mathbf{H}$ and $D = \varepsilon_0\mathbf{E}$ leads to an analogy between **B** and **D**. Let us represent the magnetic flux by Φ and define Φ as the flux passing through any designated area [25,34]:

$$\Phi = \oint_S \mathbf{B}.d\mathbf{s} \ . \tag{2.43}$$

Our analogy remind sus of the electric flux Ψ, and of coulomb's law, which states that the total flux passing through any closed surface is equal to the charge enclosed,

$$\Psi = \oint_S \mathbf{D}.d\mathbf{s} = Q \ . \tag{2.44}$$

charge Q is the source of the electric flux lines and these lines begin and terminate on positive and negative charge, respectively. No such source has ever been discovered for the lines of the magnetic flux. The magnetic flux

lines are closed and do not terminate on a 'magnetic charge'. For this reason Gauss's law for the magnetic field is,

$$\oint_S \mathbf{B}.d\mathbf{s} = 0 \ . \tag{2.45}$$

And the application of the divergence theorem gives,

$$\nabla.\mathbf{B} = 0 \ . \tag{2.46}$$

This is the last of Maxwell's four equation as they apply to static electric field and steady magnetic field [34], when collecting these equation we have for static electric fields and steady magnetic fields:

$$\nabla.\mathbf{D} = \rho_v \ . \tag{2.47}$$

$$\nabla \times \mathbf{E} = 0 \ . \tag{2.48}$$

$$\nabla \times \mathbf{H} = \mathbf{J} \ . \tag{2.49}$$

$$\nabla.\mathbf{B} = 0 \ . \tag{2.50}$$

To these four equation we may add the two expression relating **D** to **E** and **B** to **H** in free space;

$$\mathbf{D} = \varepsilon_0 \mathbf{E}$$

$$\mathbf{B} = \mu_0 \mathbf{H}$$

And also it may be helpful to add the electrostatic potential equation,

$$\mathbf{E} = -\nabla V$$

The above four equations spesify the divergence and curl of an electric and magnetic field [34]. The corresponding set of four integral equations that apply to static electric field and steady magnetic field is;

$$\oint_S \mathbf{D} \,.d\mathbf{s} = Q = \int_{vol} \rho_v dv \ . \tag{2.51}$$

$$\oint \mathbf{E}.d\mathbf{L} = 0 \ . \tag{2.52}$$

$$\oint \mathbf{H}.d\mathbf{L} = I = \int_S \mathbf{J}.d\mathbf{s} \ . \tag{2.53}$$

$$\oint_S \mathbf{B}.d\mathbf{s} = 0 \ . \tag{2.54}$$

The Scalar Electric Potential and The Vector Magnetic Potential

Let us first assume the existance of a scalar magnetic potential, which we designate V_m [34], whose negative gradient gives the magnetic field intensity,

$$\mathbf{H} = -\nabla V_m \ . \tag{2.55}$$

If we take the curl for both sides this will not conflict the previous result for the magnetic field, and therefore;

$$\nabla \times \mathbf{H} = \mathbf{J} = \nabla \times (-\nabla V_m) \ . \tag{2.56}$$

But the curl of the gradient of any scalar is zero, therefore we see that if **H** is to be defined as the gradient of a scalar magnetic potential, then the current density must be zero throughout the region in which the scalar magnetic potential is so defined,we then have :

$$\mathbf{H} = -\nabla V_m \qquad (\mathbf{J} = 0)\,. \tag{2.57}$$

We know that

$$\nabla \times \mathbf{E} = 0\,.$$

so
$$\oint \mathbf{E}.d\mathbf{L} = 0\,. \tag{2.58}$$

and therefore the line integral

$$V_{ab} = \int_b^a \mathbf{E}.d\mathbf{L}\,.$$

is independent of the path, in the magnostatic case however

$$\nabla \times \mathbf{H} = 0 \qquad (\text{wherever } \mathbf{J} = 0)\,.$$

but
$$\oint \mathbf{H}.d\mathbf{L} = I\ .$$

Even if $\mathbf{J}$ is zero along the path of integretion, if no current I is enclosed by the path, then a single valued potential function may be defined. In general however,

$$V_{m,ab} = -\int_b^a \mathbf{H}.d\mathbf{L}\,. \qquad \text{(specified path)}$$

Magnetic Forces and Inductance

A magnetic field may be produced by moving charges and may exert forces on moving charges; a magnetic field cannot arise from stationary charges and cannot exert any force on a stationary charges.

Force on a Moving Charge

We know that the force on a charged particle is given by $\mathbf{F} = Q\mathbf{E}$. **F** and **E** are in the same direction (for positive charges) [26,34], if the velocity of a charged particle is **v** , and it is moving in a magnetic field of flux density **B**, then the force's magnitude is proportional to the product of the magnitudes of the charge *Q*, its velocity **v**, and the flux density **B**. The direction of the force is perpendicular to both **v** and **B** and mathematically expressed as :

$$\mathbf{F} = \mathbf{v}Q \times \mathbf{B}. \tag{2.59}$$

Then the force on a moving particle due to combined electric and magnetic fields is obtained easily by super position

$$\mathbf{F} = Q\,(\mathbf{E}+\mathbf{v}\times\mathbf{B}). \tag{2.60}$$

This equation is known as *Lorentz force equation*.

The Magnetic Circuit

Let us begin with the electrostatic potential and its relationship to electric field intensity

$$\mathbf{E} = -\nabla V. \tag{2.61}$$

There is analogous relationship between electric field intensity and magnetic field intensity that is :

$$\mathbf{H} = -\nabla V_m . \tag{2.62}$$

In magnetic circuits we call V_m *magnetomotive force*, or mmf.

The electric potential difference between points *A* and *B* may be written as:

$$V_{AB} = \int_A^B \mathbf{E} . d\mathbf{L} . \tag{2.63}$$

And the corresponding relationship between the mmf and the magnetic field intensity :

$$V_{mAB} = \int_A^B \mathbf{H} . d\mathbf{L} . \tag{2.64}$$

Ohm's law for electric circuit has the point form: $\mathbf{J} = \sigma \mathbf{E}$, and we can see that the magnetic flux density will be the analog of the current density, $\mathbf{B} = \mu \mathbf{H}$, and then we must integrate to find the total current

$$I = \int_S \mathbf{J} . d\mathbf{s} . \tag{2.65}$$

A corresponding operation is then used to determine the total magnetic flux

$$\Phi = \int_S \mathbf{B} . d\mathbf{s} . \tag{2.66}$$

We then define the resistance as the ratio of potential difference and current, or:

$$V = IR \ . \tag{2.67}$$

We shall now define the *reluctance* as the ratio of the magnetomotive force to the total flux, thus:

$$V_m = \Phi \Re \ . \tag{2.68}$$

But $R = \frac{d}{\sigma S}$, where R is the resistance, d is the length, and S is the uniform cross section of the resistor, and we have also $\Re = \frac{d}{\mu S}$,where $\Re$ is the reluctance, d and S are as in equation above [34]. We know that the closed line integral of **E** is zero, $\oint \mathbf{E}.d\mathbf{L} = 0$.

In other words, *Kirchhoff's voltage law* states that the rise in potential through the source is exactly equal to the fall in potential through the load. The expression for magnetic phenomena takes on a slightly different form,

$$\oint \mathbf{H}.d\mathbf{L} = I_{total} \ . \tag{2.69}$$

2.5 Time-varying Fields for Maxwell's Equations

Two new concepts will now be introduced, the electric field produced by a changing magnetic field and the magnetic field produced by a changing electric field [26]. The first concept resulted from experimental research by Michael Faraday. And the second from the theoretical efforts of James Clerk Maxwell.

Faraday's Law

A time-varying magnetic field produces an electromotive force (emf) which may establish a current in a suitable closed circuit. An electromotive force is merely a voltage that arises from conductors moving in a magnetic field or from changing magnetic fields.

Faraday's Law is expressed mathematically as:

$$\text{emf} = -\frac{d\Phi}{dt} \ . \tag{2.70}$$

The emf is obviously a scalar. We define the emf as:

$\text{emf} = \oint \mathbf{E} \,.\, d\mathbf{L}$, it is the voltage about a specific closed path.

For an electric field intensity resulting from a static charge distribution must lead to zero potential difference about a closed path [34]. In electro-static, the line integral leads to a potential difference; with time-varying fields the result is an emf or a voltage, replacing Φ in eq. (2.70) by the surface integral of **B**, we have:

$$\text{emf} = \oint \mathbf{E} \,.\, d\mathbf{L} = -\frac{d}{dt} \int_S \mathbf{B} \,.\, d\mathbf{s} \ . \tag{2.71}$$

We first consider a stationary path. The magnetic flux is the only time-varying quantity on the right side of equation above, and a partial derivative may be taken under the integral sign,

$$\text{emf} = \oint \mathbf{E} \,.\, d\mathbf{L} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \,.\, d\mathbf{s} \ . \tag{2.72}$$

Now applying Stokes' theorem to the closed line integral, we have :

$$\int_S (\nabla \times \mathbf{E}) \,.\, d\mathbf{s} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \,.\, d\mathbf{s} \ . \qquad (2.73)$$

Where the surface integrals may be taken over identical surfaces, thus we obtain:

$$(\nabla \times \mathbf{E}) \,.\, d\mathbf{s} = -\frac{\partial \mathbf{B}}{\partial t} \,.\, d\mathbf{s} \ .$$

and

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \ . \qquad (2.74)$$

This is one of Maxwell's four equations as written in point form, and the eq. (2.73) is the integral form of this equation and is equivalent to Faraday's Law when applied to a fixed path. If **B** is not a function of time, then these two equations evidently reduce to the electro-static equations,

$$\oint \mathbf{E} \,.\, d\mathbf{L} = 0 \ . \qquad \text{(electrostatics)}$$

$$\nabla \times \mathbf{E} = 0 \qquad \text{(electrostatics)}$$

Displacement Current

Faraday's Law has been used to obtain Maxwell's equation (2.74) which shows that a time-changing magnetic field produces an electric field [34]. Now let us turn our attention to the time-changing electric field. We should first look from the point of Ampere's circuital law as it applies to steady magnetic fields,

$$\nabla \times \mathbf{H} = \mathbf{J}$$

Now by taking the divergence of each side,

$$\nabla.\nabla\times\mathbf{H}\equiv\mathbf{0}=\nabla.\mathbf{J}$$

The divergence of a curl is identically zero, so $\nabla\cdot\mathbf{J}$ is also zero, however the equation of continuity,

$$\nabla.\mathbf{J}=-\frac{\partial\rho_v}{\partial t}$$

which shows that

$$\nabla.\mathbf{J}=0 \text{ if } \frac{\partial\rho_v}{\partial t}=0.$$

But this is an unrealistic limitation, and the equation $\nabla\times\mathbf{H}=\mathbf{J}$ must be mended before we can accept it for time-varying fields. Suppose we add an unknown term **G** to the equation $\nabla\times\mathbf{H}=\mathbf{J}$.

Thus

$$\nabla\times\mathbf{H}=\mathbf{J}+\mathbf{G}.$$

again taking the divergence, we have :

$$0=\nabla.\mathbf{J}+\nabla.\mathbf{G}$$

Thus

$$\nabla.\mathbf{G}=\frac{\partial\rho_v}{\partial t}$$

Replacing ρ_v by $\nabla.\mathbf{D}$

We get : $$\nabla.\mathbf{G} = \frac{\partial}{\partial t}(\nabla.\mathbf{D}) = \nabla.\frac{\partial \mathbf{D}}{\partial t}$$

From which we obtain the simplest solution for **G**, $\mathbf{G} = \frac{\partial \mathbf{D}}{\partial t}$.

Ampere's circuital law in point form therefore becomes:

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \ . \qquad (2.75)$$

We now have a second one of Maxwell's equations, and we shall investigate its significance. Since the additional term $\partial \mathbf{D} / \partial t$ results from a time-varying electric flux density (or displacement density), Maxwell termed it a *displacement current density*. It is sometimes denoted by $\mathbf{J}_d$.

The eq.(2.75) then becomes;

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_d \ . \qquad (2.76)$$

$$\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t}$$

this is the third type of current density we have met [34]. Conduction current density, $\mathbf{J} = \sigma \mathbf{E}$. And the convection current density is $\mathbf{J} = \rho_v v$. In nonconducting medium **J**=0, and then we have:

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}, \qquad (\text{if } \mathbf{J}=0) \ . \qquad (2.77)$$

We now notice the symmetry between **H** and **E .** The total displacement current crossing any given surface is expressed by the surface integral

$$I_d = \int_S \mathbf{J}_d \,.d\mathbf{s} = \int_S \frac{\partial \mathbf{D}}{\partial t} \,.d\mathbf{s} \,. \tag{2.78}$$

And we may obtain the time-varying equation of Ampere's circuital law by integrating the eq.(2.75) over the surface S then we obtain

$$\int_S (\nabla \times \mathbf{H}) \,.d\mathbf{s} = \int_S \mathbf{J}.d\mathbf{s} + \int_S \frac{\partial \mathbf{D}}{\partial t}.d\mathbf{s} \,. \tag{2 .79}$$

And applying Stokes' theorem we get:

$$\oint \mathbf{H} \,.d\mathbf{L} = I + I_d = I + \int_S \frac{\partial \mathbf{D}}{\partial t}.d\mathbf{s} \,. \tag{2.80}$$

Maxwell's Equations in The Point Form

We have already obtained two of maxwell's equations for time-varying fields [26,34],

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.81}$$

and

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{2.82}$$

The remaining two equations are unchanged from their non-time-varying form:

$$\nabla . \mathbf{D} = \rho_v \tag{2.83}$$

$$\nabla . \mathbf{B} = \mathbf{0} \tag{2.84}$$

Equation (2.83) states that charge density is a source (or sink) of electric flux lines. Note that we can no longer say that all electric flux begins and terminates on charge. Because the point form of Faraday's law (2.81)shows that **E** and hence **D,** may have circulation if a changing magnetic field is present. Thus the lines of the electric flux may form closed loops. However, the converse is still true, and every coulomb of charge must have one coulomb of electric flux diverging from it. Equation (2.84) again acknowledges the fact that "magnetic charges" are not known to exist. Magnetic flux is always found in closed loops and never diverges from a point source.

These four equations form the basis of all electromagnetic theory. They are partial differential equations and relate the electric and magnetic fields to each other and to their sources, charge and current density. The auxiliary equations relating **D** and **E**:

$$\mathbf{D} = \varepsilon \mathbf{E}$$

Relating **B** and **H**,

$$\mathbf{B} = \mu \mathbf{H}$$

Defining conduction current density, $\mathbf{J} = \sigma \mathbf{E}$

Defining convection current density in terms of the volume charge density ρ_v , $\mathbf{J} = \rho_v \mathbf{v}$; are also required to define and relate the quantities appearing in Maxwell's equations.

Finally, because of its fundamental importance we should include the Lorentz Force equation, written in its point form as the force per unit volume,

$$\mathbf{F} = \rho_v (\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Maxwell's equations in The Integral Form

The integral forms of Maxwell's equations are usually easier to recognize in terms of the experimental laws from which they have been obtained by a generalization process. Experiments must treat physical macroscopic quantities, and their results therefore are expressed in terms of integral relationship. A differential equation always represents a theory. Let us now collect the integral forms of Maxwell's equations of the previous section [34]. Integrating (2.81) over a surface and applying Stokes' theorem, we obtain Faraday's law,

$$\oint \mathbf{E} . d\mathbf{L} = -\int_S \frac{\partial \mathbf{B}}{\partial t} . d\mathbf{s} \ . \tag{2.85}$$

And the same process is applied to (2.82) yields Ampere's circuital law,

$$\oint \mathbf{H} . d\mathbf{L} = I \ + \int_S \frac{\partial \mathbf{D}}{\partial t} . d\mathbf{s} \ . \tag{2.86}$$

Gausse's law for the electric and magnetic fields are obtained by integrating (2.83) & (2.84) throughout a volume and using the divergence theorem:

$$\oint_S \mathbf{D} . d\mathbf{s} = \int_{vol} \rho_v dv \ . \tag{2.87}$$

$$\oint_S \mathbf{B} . d\mathbf{s} = 0 . \tag{2.88}$$

These four integral equations enable us to find the boundary conditions on **B**, **D**, **H** and **E** which are necessary to evaluate the constants obtained in solving Maxwell's equations in partial differential form. These boundary conditions are in general unchanged from their forms for static or steady fields. And the same methods may be used to obtain them. It is always desirable to idealize a physical problem by assuming a perfect conductor for which σ is infinite, but **J** is finite. From Ohm's law, then, in a perfect conductor $\mathbf{E}=0$, and it follows from the point form of Faraday's law that $\mathbf{H}=0$. For time-varying fields; the point form of Ampere's circuital law then shows that the finite volume of **J** is $\mathbf{J}=0$.

The Related Potentials

The time-varying potentials, usually called *retarded* potentials [26], we may summerize the use of potentials by stating that a knowledge of the distribution of ρ_v and **J** throughout space theoretically enables us to determine V and **A** from:

$$V = \int_{vol} \frac{[\rho_v]}{4\pi\varepsilon R} dv \quad . \tag{2.89}$$

and

$$\mathrm{A} = \int_{vol} \frac{\mu[\mathbf{J}]}{4\pi R} dv \quad . \tag{2.90}$$

The electric and magnetic fields are then obtained . If the charge and current distribution are unknown ,or reasonable approximations cannot be

made for them,these potentials usually offer no easier path toward the solution than does the direct application of Maxwell's equations.

2.6 Time-Harmonic Fields for Maxwell's Equations

Up to this point,we have considered the general case of arbitrary time variation of EM fields. In many practical situations, especially at low frequencies, it is sufficient to deal with only the steady-state (or equilibrium) solution of EM fields when produced by sinusoidal currents. Such fields are said to be sinusoidal time-varying or time harmonic [26], that is, they vary at a sinusoidal frequency ω. An arbitrary time-dependent field $\mathbf{F}(x, y, z, t)$ or $\mathbf{F}(\mathbf{r},t)$ can be expressed as,

$$\mathbf{F}(\mathbf{r}, t) = \mathrm{Re}\left[\mathbf{F}_s(\mathbf{r}) e^{j\omega t}\right] \tag{2.91}$$

Where $\mathbf{F}_s(\mathbf{r}) = \mathbf{F}_s(x, y, z)$ is the phasor form of $\mathbf{F}(\mathbf{r},t)$ and is in general complex, Re[] indicates "taking the real part of" quantity in brackets, and ω is the angular frequency (in rad/s) of the sinusoidal excitation. The EM field quantities can be represented in phasor notation as

$$\begin{bmatrix} \mathbf{E}(\mathbf{r},t) \\ \mathbf{D}(\mathbf{r},t) \\ \mathbf{H}(\mathbf{r},t) \\ \mathbf{B}(\mathbf{r},t) \end{bmatrix} = \begin{bmatrix} \mathbf{E}_s(\mathbf{r}) \\ \mathbf{D}_s(\mathbf{r}) \\ \mathbf{H}_s(\mathbf{r}) \\ \mathbf{B}_s(\mathbf{r}) \end{bmatrix} e^{j\omega t} \tag{2.92}$$

Using the phasor representation allows us to replace the time derivations $\partial/\partial t$ by $j\omega$ since

$$\frac{\partial e^{j\omega t}}{\partial t} = j\omega e^{j\omega t} \tag{2.93}$$

Thus Maxwell's equations, in sinusoidal steady state, become

$$\nabla \cdot \mathbf{D}_s = \rho_{vs} \tag{2.94}$$

$$\nabla \cdot \mathbf{B}_s = 0 \tag{2.95}$$

$$\nabla \times \mathbf{E}_s = -j\omega \mathbf{B}_s - \mathbf{J}_{ms} \tag{2.96}$$

$$\nabla \times \mathbf{H}_s = \mathbf{J}_{es} + j\omega \mathbf{D}_s \tag{2.97}$$

We should observe that the effect of the time-harmonic assumption is to eliminate the time dependence from Maxwell's equations, thereby reducing the time-space dependence to space dependence only. A non-sinusoidal field can be represented as

$$\mathbf{F}(\mathbf{r},t) = \mathrm{Re}\left[\int_{-\infty}^{\infty} \mathbf{F}_s(\mathbf{r},\omega) e^{j\omega t} d\omega\right] \tag{2.98}$$

Thus the solutions to Maxwell's equations for a nonsinusoidal field can be obtained by summing all the Fourier components $\mathbf{F}_s(\mathbf{r},\omega)$ over ω. Hence forth, we drop the subscript s denoting phasor quantity when no confusion results. Replacing the time derivative in the wave equation

$$\nabla^2 \Psi - \frac{1}{u^2} \frac{\partial^2 \Psi}{\partial t^2} = 0 \tag{2.99}$$

By $(j\omega)^2$ yields the scalar wave equation in phasor representation as

$$\nabla^2 \Psi + k^2 \Psi = 0 \tag{2.100}$$

Where k is the propagation constant (in *rad/m*), given by

$$k = \frac{\omega}{u} = \frac{2\pi f}{u} = \frac{2\pi}{\lambda} \tag{2.101}$$

We recall that the wave equation for the electric and magnetic field respectively are

$$\nabla^2 \mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \tag{2.102}$$

$$\nabla^2 \mathbf{H} - \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \tag{2.103}$$

which is the time-dependent wave equation. The velocity (in m/s) of wave propagation is $u = \frac{1}{\sqrt{\mu\varepsilon}}$

Where $u = c \approx 3\times 10^8$ m/s in free space. These two equations were obtained assuming that $\rho_v = 0 = \mathbf{J}$. If $\rho_v \neq 0 \neq \mathbf{J}$, Eq. (2.100) will have the general form

$$\nabla^2 \Psi + k^2 \Psi = g \tag{2.104}$$

We notice that this Helmholtz equation reduces to:

(1) Poisson's equation

$$\nabla^2 \Psi = g \tag{2.105}$$

When $k = 0$ (i.e., $\omega = 0$ for static case).

(2) Laplace's equation

$$\nabla^2\Psi = 0 \tag{2.106}$$

When $k = 0 = g$.

Thus Poisson's and Laplace's equations are special cases of the Helmholtz equation. Note that function Ψ is said to be harmonic if it satisfies Laplace's equation.

2.7 The Relation Between Maxwell's Equations and The Wave Equation

As we have mentioned above, Maxwell's equations are coupled first-order differential equations which are difficult to apply when solving boundary-value problems. The difficulty is overcome by decoupling the first-order equations [26], thereby obtaining the wave equation which is a second-order differential equation is more efficient and useful for solving problems. In section 2.6 we finally ended up with Maxwell's equations, the four equations which encapsulate everything we know about electricity and magnetism. These equations are:

$$\nabla . \mathbf{E} = 4\pi\rho \tag{2.107}$$

$$\nabla . \mathbf{B} = 0 \tag{2.108}$$

$$\nabla \times \mathbf{E} = -\frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t} \tag{2.109}$$

or $$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\,\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \tag{2.110}$$

To obtain the wave equation for a linear, isotropic, homogeneous, source-free medium, then setting $\rho = 0$ and $\mathbf{J} = 0$ so we obtain :

$$\nabla . \mathbf{E} = 0 \tag{2.111}$$

$$\nabla . \mathbf{B} = 0 \tag{2.112}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{2.113}$$

$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \ . \tag{2.114}$$

variations in $\mathbf{E}$ act as a source for $\mathbf{B}$, which in turn acts as a source for $\mathbf{E}$, which in turn acts as a source for $\mathbf{B}$, which ... The goal we need now is to fully understand this coupled behavior. To do so, we will find it easiest to first uncouple these equations. We do this by taking the curl of each equation. Let's begin by taking the curl of equation (2.109).

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times \left(-\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \right) .$$

The curl of the left-hand side of this equation is

$$\nabla\times\nabla\times\mathbf{E}=\nabla\left(\nabla\cdot\mathbf{E}\right)-\nabla^2\mathbf{E}=-\nabla^2\mathbf{E}$$

The simplification follows because we have restricted ourselves to the source free equations $\rho=0$ and so $\nabla\cdot\mathbf{E}=0$. Now, look at the curl of the right-hand side:

$$\nabla\times\left(-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\right)=-\frac{1}{c}\frac{\partial}{\partial t}\left(\nabla\times\mathbf{B}\right)=-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}$$

Putting the left and right sides together, we end up with

$$\frac{\partial^2\mathbf{E}}{\partial t^2}-c^2\nabla^2\mathbf{E}=0 \tag{2.115}$$

Repeating this procedure for the other equations, we end up with something that is essentially identical, but for the magnetic field:

$$\frac{\partial^2\mathbf{B}}{\partial t^2}-c^2\nabla^2\mathbf{B}=0 \tag{2.116}$$

We will simplify things initially by imagining that $\mathbf{E}$ only depends on x and t . The equation we derived for $\mathbf{E}$ then reduces to

$$\frac{\partial^2\mathbf{E}}{\partial t^2}-c^2\frac{\partial^2\mathbf{E}}{\partial x^2}=0 \tag{2.117}$$

which is the time-dependent vector Helmholtz equation or simply wave equation. If we had started the derivation with Eq.(2.82) we would obtain the wave equation for $\mathbf{H}$ as

$$\nabla^2\mathbf{H}-c^2\frac{\partial^2\mathbf{H}}{\partial t^2}=0 \tag{2.118}$$

It should be noted that each of the vector equations in (2.116) and (2.118) has three scalar components, so that altogether we have six scalar equations for E_x, E_y, E_z H_x, H_y, and H_z. Thus each component of the wave equations has the form

$$\nabla^2\Psi - \frac{1}{u^2}\frac{\partial^2\Psi}{\partial t^2} = 0 \tag{2.119}$$

The velocity (in m/s) of wave propagation is

$$u = \frac{1}{\sqrt{\mu\varepsilon}}$$

Where $u = c \approx 3\times10^8$ m/s in free space.

2.8 The Relation Between Maxwell's Equations and The Potentials Functions

Although we are often interested in electric and magnetic field intensities ($\mathbf{E}$ and $\mathbf{H}$), which are physically measurable quantities, it is often convenient to use auxiliary functions in analyzing an EM field. These auxiliary functions are the scalar electric potential V and vector magnetic potential $\mathbf{A}$. Although these potential functions are arbitrary, they are required to satisfy Maxwell's equations. Their derivation is based on two fundamental vector identities

$$\nabla\times\nabla\Phi = 0 \tag{2.120}$$

and

$$\nabla\cdot\nabla\times\mathbf{F} = 0 \tag{2.121}$$

which an arbitrary scalar field Φ and vector field $\mathbf{F}$ must satisfy. Maxwell's equation (2.112) along with equation (2.121) is satisfied if we define $\mathbf{A}$ such that

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{2.122}$$

Substituting this into Eq. (2.109) gives

$$-\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \tag{2.123}$$

Since this equation has to be compatible with equation (2.120) we can choose the scalar field V such that

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V \tag{2.124}$$

or

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \tag{2.125}$$

Thus, if we knew the potential functions V and $\mathbf{A}$, the fields $\mathbf{E}$ and $\mathbf{B}$ could be obtained from Eqs. (2.122) and (2.124). However, we still need to find the solution for the potential functions. Substituting Eqs. (2.122) and (2.124) into Eq. (2.82) and assuming a linear, homogeneous medium, and make the substitutions required we obtain:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon} = -\nabla^2 V - \frac{\partial(\nabla \cdot \mathbf{A})}{\partial t} \tag{2.126}$$

or $$\nabla^2 V + \frac{\partial}{\partial t}\nabla \cdot \mathbf{A} = -\frac{\rho}{\varepsilon} \tag{2.127}$$

According to the Helmholtz theorem of vector analysis, a vector is uniquely defined if and only if both its curl and divergence are specified. We have only specified the curl of $\mathbf{A}$ in Eq. (2.122); we may choose the divergence of $\mathbf{A}$ so that the differential equations (2.124) and (2.125) have the simplest forms possible. We achieve this in the so-called *Lorentz condition*:

$$\nabla \cdot \mathbf{A} = -\mu\varepsilon\frac{\partial V}{\partial t} \tag{2.128}$$

Incorporating this condition into Eqs. (2.124) and (2.125) results in

$$\nabla^2 \mathbf{A} - \mu\varepsilon\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu\mathbf{J} \tag{2.129}$$

and

$$\nabla^2 V - \mu\varepsilon\frac{\partial^2 V}{\partial t^2} = -\frac{\rho_v}{\varepsilon} \tag{2.130}$$

which are inhomogeneous wave equations. Thus Maxwell's equations in terms of the potentials V and $\mathbf{A}$ reduce to the three equations (2.128), (2.129), (2.130). In other words, the three equations are equivalent to the ordinary form of Maxwell's potentials satisfying these equations always lead to a solution of Maxwell's equations for $\mathbf{E}$ and $\mathbf{B}$ when used with

Eqs. (2.122) and (2.125). Without proofs the integral solutions to Eqs. (2.129) and (2.130) are the so-called *retarded* potentials

$$V = \int_{vol} \frac{[\rho_v]}{4\pi\varepsilon R} dv \quad .$$

and

$$\mathrm{A} = \int_{vol} \frac{\mu[\mathbf{J}]}{4\pi R} dv \quad .$$

Where R is the distance from the source point to the field point, and the square brackets denote ρ_v and $\mathbf{J}$ are specified at a time $R(\mu\varepsilon)^{1/2}$ earlier than for which $\mathbf{A}$ or V is being determined.

Chapter Three

Analytical Methods for Maxwell's Equations

3. Analytical Methods for Maxwell's Equations

3.1 Introductionto Analytical Methods

The analytical methods provide the most satisfactory solution for the Maxwell's equations. However, the range of problems which can be solved using the analytical methods are very much limited [26]. The reasons for this limitation are: irregular shape of the structure, dielectric inhomogeneity, and/or inhomogeneous boundary conditions. Therefore, approximation methods, computational methods or any other methods are employed in such situations. The analytical solutions, although limited, are useful in validating the results of the computational methods. Also, one is able to appreciate the need for computational methods better after seeing the limitations of other methods. The general methods for solving the Maxwell's equations may be classified into two broad categories: analytical methods and computational methods. Some of the analytical methods are listed below.

Analytical Methods (Exact Solutions).

(1) Method of Separation of Variables: Here the PDE is split into ordinary differential equations that may be solved easily. This analytical method may not always work, but it is often the simplest when it does work .

(2) Eigen-Function Expansion Method: We use this method to solve non-homogeneous problems which could not be solved by the previous method.

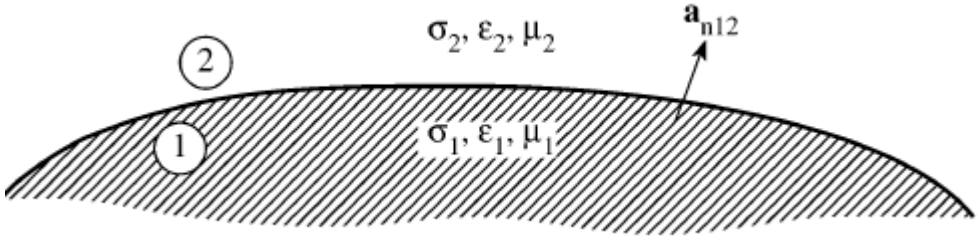
(3) Green's Function: This analytical method produces a solution in the form of an integral.

(4) Conformal Mapping: This method is limited to solving the Laplace equation in two dimensions .

(5) Integral Transforms: May be used to solve the Maxwell's equations. This involves Laplace and Fourier transforms methods .

3.2 Interface and Boundary Conditions

The material medium in which an EM field exists is usually characterized by its constitutive parameters σ, ε, and μ. The medium is said to be *linear* if σ, ε, and μ are independent of **E** and **H** or nonlinear otherwise. It is *homogeneous* if σ, ε, and μ are not functions of space variables or inhomogeneous otherwise [26]. It is *isotropic* if $\sigma, \varepsilon,$ and μ are independent of direction (scalars) or anisotropic otherwise.



The electromagnetic fields obtained from the solution of Maxwell's equations must also satisfy the boundary conditions at the interface between different media. As it turns out, that the boundary conditions for time-varying fields are exactly the same as those for static fields. We state these boundary conditions without their proofs.

These conditions are summarized as follows [15] :

Scalar Form	Vector Form	
$E_{t1} = E_{t2}$	$\mathbf{a}_n \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$	(3.1)
$H_{t1} - H_{t2} = J_s$	$\mathbf{a}_n \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$	(3.2)
$B_{n1} = B_{n2}$	$\mathbf{a}_n \times (\mathbf{B}_1 - \mathbf{B}_2) = 0$	(3.3)
$D_{n1} - D_{n2} = \rho_s$	$\mathbf{a}_n \bullet (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$	(3.4)
$J_{n1} = J_{n2}$	$\mathbf{a}_n \bullet (\mathbf{J}_1 - \mathbf{J}_2) = 0$	(3.5)
$\frac{J_{t1}}{\sigma_1} = \frac{J_{t2}}{\sigma_2}$	$\mathbf{a}_n \times \left[\frac{\mathbf{J}_1}{\sigma_1} - \frac{\mathbf{J}_2}{\sigma_2} \right] = 0$	(3.6)

The subscripts t_1 and t_2 refer to components of fields tangential to the boundary in media 1 and 2, respectively. Similarly, the subscripts n_1 and n_2 indicate the field components normal to the boundary. The unit vector $\mathbf{a}_n$ at the interface points into medium 1, ρ_s is the free surface charge density, and $\mathbf{J}_s$ is the free surface current density. Equation (3.1) states that the tangential components of $\mathbf{E}_1$ and $\mathbf{E}_2$ are equal at the interface (boundary). However (3.2) asserts that the tangential components of $\mathbf{H}_1$ and $\mathbf{H}_2$ at any point on the interface are discontinuous by an amount equal to the surface current density at that point. Equation (3.3) states that the normal components of $\mathbf{B}_1$ and $\mathbf{B}_2$ at the interface are continuous. Hower , (3.4) states that the normal components of $\mathbf{D}_1$ and $\mathbf{D}_2$ are discontinuous at any point on the interface by an amount equal to the surface charge

density at that point. Equation (3.5) states that the normal components of $\mathbf{J}_1$ and $\mathbf{J}_2$ are equal at the interface. Equation (3.9) states that the ratio of the tangential components of current densities at the interface is equal to the ratio of the conductivities.

When applying the boundary conditions, we must keep in mind the followings:

(1) The electromagnetic fields inside a perfect conductor $(\sigma = \infty)$ are zero. Thus, on the surface of a perfect conductor, both ρ_s and $\mathbf{J}_s$ can exist.

(2) Time-varying fields can exist inside a conductor $(\sigma < \infty)$. Hence, $\mathbf{J}_s$ is zero, but ρ_s can exist at the boundary between a conductor and a perfect dielectric.

(3) At the interface between two perfect dielectrics $\mathbf{J}_s$ is zero. However, ρ_s is zero unless the charge is physically placed at the interface.

Electromagnetic fields existing in any medium must satisfy Maxwell's equations. When we seek solutions of Maxwell's equations in two or more media, we must ascertain that the fields are matched at the boundaries.

3.3 General wave Equation

Consider a uniform but source-free medium having dielectric constant ε, magnetic permeability μ, and conductivity σ. The medium is source free as long as it does not contain the charges and currents necessary to generate the fields. The conduction current density can exist as determined by Ohm's

Law ($\mathbf{J}=\sigma\mathbf{E}$) in a finitely conducting medium. Under these conditions, Maxwell's equations are [15]:

$$\nabla\times\mathbf{E}=-\mu\frac{\partial\mathbf{H}}{\partial t}, \tag{3.7}$$

$$\nabla\times\mathbf{H}=\sigma\mathbf{E}+\varepsilon\frac{\partial\mathbf{E}}{\partial t}, \tag{3.8}$$

$$\nabla\cdot\mathbf{B}=0\Rightarrow\nabla\cdot\mathbf{H}=0 \tag{3.9}$$

$$\nabla\cdot\mathbf{D}=0\Rightarrow\nabla\cdot\mathbf{E}=0 \tag{3.10}$$

Where

$$\mathbf{B}=\mu\mathbf{H},\ \text{and}\ \ \mathbf{D}=\varepsilon\mathbf{E}\ .$$

For *linear* ($\mathbf{D}$ is parallel to $\mathbf{E}$, and $\mathbf{B}$ is parallel to $\mathbf{H}$), *homogeneous* (medium properties are the same at all points), and isotropic (μ and ε are independent of direction) medium, both μ and ε are scalar constant. Such a medium is called ***uniform medium***. The preceding equations are in terms of two variables, let us now obtain an equation in terms of one variable. Say the electric field $\mathbf{E}$ only, to do this we take the curl of eq.(3.7) and obtain

$$\nabla\times\nabla\times\mathbf{E}=-\mu\nabla\times\left(\frac{\partial\mathbf{H}}{\partial t}\right). \tag{3.11}$$

Using the vector identity

$$\nabla\times\nabla\times\mathbf{E}=\nabla(\nabla.\mathbf{E})-\nabla^2\mathbf{E}\ .$$

and substituting

$$\nabla.\mathbf{E} = 0 \ ,$$

we have

$$\nabla \times \nabla \times \mathbf{E} = -\nabla^2 \mathbf{E} \ .$$

where the Laplacian of a vector quantity is defined in the rectangular coordinate system as

$$\nabla^2 \mathbf{E} = \nabla^2 E_x \mathbf{a}_x + \nabla^2 E_y \mathbf{a}_y + \nabla^2 E_z \mathbf{a}_z \ . \tag{3.12}$$

and the Laplacian operator is

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \ . \tag{3.13}$$

Now changing the order of differentiation with respect to space and time, we can write (3.11) as

$$\nabla^2 \mathbf{E} = \mu \frac{\partial}{\partial t} \left[\nabla \times \mathbf{H} \right] \ . \tag{3.14}$$

Setting equ. (3.8) into equ.(3.14), we get

$$\nabla^2 \mathbf{E} = \mu\sigma \frac{\partial \mathbf{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} \ . \tag{3.15}$$

We can also obtain a similar equation in terms of the $\mathbf{H}$ field as

$$\nabla^2 \mathbf{H} = \mu\sigma \frac{\partial \mathbf{H}}{\partial t} + \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} \ . \tag{3.16}$$

The set of six independent equations given by (3.15) and (3.16) are known as the general wave equations. These equations govern the behavior of all

electromagnetic fields in a uniform but source-free conducting medium. The presence of the first-order term in a second-order differential equation indicates that the fields decay (lose energy) as they propagate through the medium. For this reason, a conducting medium is called a *lossy-medium*. In the next section we will solve these equations and show that each equation does, in fact, represent an electromagnetic wave.

3.4 Plane Wave in Dielectric Medium

Let us consider a dielectric medium in which the conduction current is almost nonexistent in comparison with the displacement current [15], such a medium is called ***perfect dielectric*** or ***lossless medium*** $(\sigma = 0)$. Thus, by setting $(\sigma = 0)$ in (3.15) and (3.16), we obtain the wave equation for a lossless medium as

$$\nabla^2 \mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 , \tag{3.17}$$

and

$$\nabla^2 \mathbf{H} - \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0. \tag{3.18}$$

These equations are called the time-dependent *Helmholtz equations*. The absence of the first-order term signifies that the electromagnetic fields do not decay (not losing energy) as they propagate in a lossless medium. The velocity of wave propagation is $c = \frac{1}{\sqrt{\mu\varepsilon}}$ where $c \approx 3\times 10^8$ m/s. Thus, the wave equation in general has the form

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \ . \tag{3.19}$$

Also one can show that the time-varying potentials equations are as follows

$$\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} \ , \tag{3.20}$$

and

$$\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho_v}{\varepsilon} \ , \tag{3.21}$$

which are inhomogeneous wave equations.

3.5 Separation of Variables Method

The method of separation of variables (sometimes called the method of Fourier) is a convenient method for solving the Maxwell's equation. Basically, it entails seeking a solution which breaks up into a product of functions, each of which involves only one of the variables [26]. For example, in two variables, The solution $u(x,y)$ can be written as:

$$u(x,y) = X(x)\,Y(y)$$

This separates out the partial differential equation into two or three ordinary differential equations which are related by a common constant (usually Eigen-Values).

3.5.1 One Dimensional Wave Equation

We begin the application of separation of variables by finding the product solution of the homogeneous scalar one dimensional wave equation [4,26], which is special case of Maxwell's equations,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Subject to the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0 \text{ for all } t > 0$$

and the initial conditions $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$ for $0 < x < L$.

One can show that the general solution to this initial boundary value problem

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi}{L}x \, (A_n \cos\lambda_n t + B_n \sin\lambda_n t)$$

where

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x \, dx \quad , \quad B_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi}{L}x \, dx$$

and

$$\lambda_n = c\frac{n\pi}{L}, \quad n = 1,2,\ldots.$$

3.5.2 Two Dimensional Wave Equation

Now we consider the two dimensional wave equation [4,26],

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad 0 < x < a\ , \quad 0 < y < b\ , \quad t > 0$$

Where $u = u(x, y, t)$, subject to the boundary conditions

$$u(0, y, t) = 0 \text{ and } u(a, y, t) = 0 \text{ for } 0 \le y \le b \text{ and } t \ge 0$$

$$u(x, 0, t) = 0 \text{ and } u(x, b, t) = 0 \text{ for } 0 \le x \le a \text{ and } t \ge 0$$

and the initial condition

$$u(x, y, 0) = f(x, y) \quad \text{and} \quad \frac{\partial u}{\partial t}(x, y, 0) = g(x, y)$$

With the help of separation of variables,

$$u(x, y, t) = X(x)\, Y(y)\, T(t)$$

one can show that the solution is

$$u(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(A_{mn}\cos\lambda_{mn}t + B_{mn}\sin\lambda_{mn}t)\sin\frac{m\pi}{a}x\ \sin\frac{n\pi}{b}y$$

where $$A_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x, y)\sin\frac{m\pi}{a}x\ \sin\frac{n\pi}{b}y\ dx\ dy$$

$$B_{mn} = \frac{4}{ab\lambda_{mn}}\int_0^b\int_0^a g(x, y)\sin\frac{m\pi}{a}x\ \sin\frac{n\pi}{b}y\ dx\ dy$$

and $\lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}$.

3.5.3 Two-Dimensional Wave Equation in Polar Coordinates

Coordinate geometries other than rectangular Cartesian are used to describe many EM problems whenever it is necessary and convenient. For example, a problem having cylindrical symmetry is best solved in cylindrical system.

It is sometimes called "cylindrical polar coordinates" and " polar cylindrical coordinates. electromagnetic fields produced by an electric current in a long, straight wire is an example of it. The two dimensional wave equation in polar coordinates now is given by:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right).$$

Here we will assume that the solution depends only on the radius r and does not depend on θ, that is $\partial u/\partial\theta = 0$, then the equation becomes

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right), \qquad (3.22)$$

Where $u = u(r,t)$, and $0 < r < a,$ and $t > 0$,

with the boundary condition $u(a, t) = 0,\ t \geq 0$,

and initial conditions are $u(r, 0) = f\ (r)$, $\frac{\partial u}{\partial t}(r,0) = g\,(r),\ 0 < r < a$.

we will solve this boundary value problem using separation of variables method, and reduce the problem into two ordinary differential equations in

r and t . Hence the solution in t will consist of sins and cosines [4], the equation in the spatial variable r is new , and its solution will involve the *Bessel-Functions*. We assume that the solution is of the form $u(r,t)=R(r)T(t)$.. After differentiating and plugging into equation (3.22) and separating the variables we get

$$r^2R''(r)+rR'(r)+\lambda^2r^2R(r)=0\ ,\ R(a)=0. \tag{3.23}$$

$$T''(t)+c^2\lambda^2T(t)=0. \tag{3.24}$$

Equation (3.23) is known parametric form of Bessel's equation of order zero (here λ is the parameter), since the equation is second order and homogeneous, we need only two linearly independent solutions to be able to write its general solution. By convention, these two linearly independent solutions are called Bessel functions of the first and second kind. And are denoted by $J_0(\lambda r)$ and $Y_0(\lambda r)$, respectively. Hence the general solution to the parametric form of Bessel's equation is

$$R(r)=c_1J_0(\lambda r)+c_2Y_0(\lambda r), \qquad \text{where } r>0$$

Since on physical grounds the solutions to the wave equation are expected to be bounded, It follows that the spatial part of the solution, $R(r)$, has to be bounded near $r=0.$ This is effectively a second boundary condition on R . Now the fact that Y_0 is unbounded near 0 forces us to choose $c_2=0$ in (3.22). To avoid trivial solution, we will take $c_1=1$ and get $R(r)=J_0(\lambda r)$ the solution, $R(a)=0$, implies that $J_0(\lambda a)=0$, and so λa

must be a root of the Bessel function J_0 , this suggests J_0 has infinitely many positive zeros, which we denote by $\alpha_1 < \alpha_2 < ... < \alpha_n < ...$

Thus, $\lambda = \lambda_n = \frac{\alpha_n}{a},\ n = 1, 2, ...$

and the corresponding solutions are

$$R_n(r) = J_0(\frac{\alpha_n}{a} r),\quad n=1,2,...$$

Where α_n is the nth positive zero of J_0. And for the T equation we get

$$T(t) = T_n(t) = A_n \cos c\lambda_n t + B_n \sin c\lambda_n t\ .$$

We thus obtain the product solution

$$u(r,t) = \sum_{n=1}^{\infty} (A_n \cos c\lambda_n t + B_n \sin c\lambda_n t)\, J_0(\lambda_n r)\quad .$$

We determine the unknown coefficients by evaluating the series at $t = 0$, and using the initial condition, thus we get

$$u(r,0) = f(r) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r),\ 0 < r < a\ .$$

This series representation of $f(r)$ is known as Bessel or Fourier-Bessel expansion, thus the Bessel coefficients A_n are given by

$$A_n = \frac{2}{a^2 J_1^2(\alpha_n)} \int_0^a f(r) J_0(\lambda_n r)\, r\, dr$$

where J_1 is the Bessel function of order 1. Now differentiating the series for u with respect to t, and then setting $t = 0$, we get from the second initial condition

$$u_t(r,0) = g(r) = \sum_{n=1}^{\infty} c\lambda_n B_n \; J_0(\lambda_n r) \; .$$

Thus, $$c\lambda_n B_n = c\frac{\alpha_n}{a} B_n$$

is the nth Bessel coefficient of g and so

$$B_n = \frac{2}{c\;\alpha_n a J_1^2(\alpha_n)} \int_0^a g(r) J_0(\lambda_n r) r \, dr$$

this completely determines the solution.

Thus the solution of the radially symmetric two-dimensional wave equation with the boundary and the initial conditions is

$$u(r,t) = \sum_{n=1}^{\infty} (A_n \cos c\lambda_n t + B_n \sin c\lambda_n t) J_0(\lambda_n r)$$

where $$A_n = \frac{2}{a^2 J_1^2(\alpha_n)} \int_0^a f\,(r) J_0(\lambda_n r) r \, dr$$

$$B_n = \frac{2}{c\;\alpha_n a J_1^2(\alpha_n)} \int_0^a g(r) J_0(\lambda_n r) r \, dr \; .$$

and $\lambda_n = \frac{\alpha_n}{a}$ and α_n is the nth positive zero of J_0.

3.5.4 Laplace Equation in Spherical Coordinates

Spherical coordinates are (r,θ,ϕ); where $0 \le r \le \infty$; $0 \le \theta \le \pi$; $0 \le \phi \le 2\pi$. In this system, the wave equation becomes

$$\nabla^2 U + k^2 U = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial v}{\partial r}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial v}{\partial \theta}) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 v}{\partial \phi^2} + k^2 v = 0 .$$

Here we will solve Laplace's equation in two dimensions as it is related to potential theory. This is often written as $\Delta\varphi = 0$ or $\nabla^2\varphi = 0$, where $\Delta = \nabla^2$ is the Laplace operator and φ is a scalar function. The general theory of solutions to Laplace's equation is known as potential theory [4,26].

Consider a spherical symmetric system. If we want to solve Laplace's equation it is natural to use spherical coordinates. Assuming that the system has azimuthal symmetry $\left(\frac{\partial v}{\partial \varphi} = 0\right)$. Let us consider the problem of finding the potential distribution due to an uncharged conducting sphere of radius *r* located in an external uniform electric field as then Laplace's equation reads:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial v}{\partial r}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial v}{\partial \theta}) = 0 .$$

Multiplying both sides by r^2 we obtain :

$$\frac{\partial}{\partial r}(r^2\frac{\partial v}{\partial r}) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial v}{\partial \theta}) = 0.$$

Consider the possibility that the general solution of the equation is the product of a function $R(r)$ which depends only on the distance r, and a function $\theta(\theta)$, which depends only on the angle θ.

$$V(r,\theta) = R(r)\,\theta(\theta) .$$

Substituting this "solution" into Laplace's equation and dividing each term of this equation by $R(r)\,\theta(\theta)$ we obtain :

$$\frac{1}{R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}R(r)\right) + \frac{1}{\theta(\theta)}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\theta(\theta)\right) = 0$$

The first term of this expression depends only on the distance r while the second term depends only on the angle θ. This equation can only be true for all r and θ if :

$$\frac{1}{R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}R(r)\right) = m(m+1) = \text{constant}$$

and

$$\frac{1}{\theta(\theta)}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\theta(\theta)\right) = -m(m+1) .$$

Consider a solution for R of the following forms:

$$R(r) = A\,r^k .$$

Where A and k are arbitrary constants. Substituting this expression in the differential equation for $R(r)$ we obtain :

$$\frac{1}{Ar^k}\frac{\partial}{\partial r}(k\,r^2 A\,r^{k-1}) = \frac{k}{r^k}(k+1) = m(m+1) .$$

Therefore, the constant k must satisfy the following relation :

$$k(k+1) = k^2 + k = m(m+1) .$$

This equation gives us the following expression for k

$$k = 1 \mp \frac{\sqrt{1+4m(m+1)}}{2} = \frac{-1 \mp 2(m+\frac{1}{2})}{2} = m \text{ or } -(m+1)$$

The general solution for $R(r)$ is thus given by:

$$R(r) = A\, r^k + \frac{B}{r^{k+1}}$$

where A and B are arbitrary constants. The angle dependent part of the solution of Laplace's equation must satisfy the following equation:

$$\frac{\partial}{\partial\theta}(\sin\theta \frac{\partial}{\partial\theta}\theta(\theta)) + m(m+1)\,\theta(\theta)\sin\theta = 0 .$$

The solution of this equation is known as the *Legendre Polynomial* $P_m(\cos\theta)$. Associated Legendre polynomials are the most general solution to the Legendre's Equation and Legendre polynomials [4] are solutions that are azimuthally symmetric. In mathematics, Legendre functions are solutions to Legendre's differential equation:

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n(x)\right]+n(n+1)P_n(x)=0\ .$$

These solutions for $n=0,1,2,...$ (with the normalization $P_n(1)=1$ form a polynomial sequence of orthogonal polynomials called the Legendre polynomials. Each Legendre polynomial $P_n(x)$ is an nth degree polynomial. It may be expressed using *Rodrigues' formula:*

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right].$$

The first few Legendre polynomials are

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2-1)$

Combining the solutions for $R(r)$ and $\theta(\theta)$ we obtain the most general solution of Laplace's equation in a spherical symmetric system as:

$$V\ (r,\theta)=\sum_{m=0}^{\infty}(A_m r^m+\frac{B_m}{r^{m+1}})P_m(\cos\theta)\ .$$

If the potential at the surface of a sphere is given by

$$V_0(\theta)=k\cos(3\theta)$$

where k is some constant. Find the potential inside and outside the sphere, as well as the sphere charge density $\sigma(\theta)$ is on the sphere. (Assume that there is no charge inside or outside the sphere).

The general solution of Laplace's equation in spherical coordinates as we have got above is

$$V(r,\theta)=\sum_{m=0}^{\infty}(A_m r^m+\frac{B_m}{r^{m+1}})P_m(\cos\theta)\ .$$

First: consider the region inside the sphere where $(r<R)$, in this region $B_m=0$, since otherwise $V(r,\theta)$ would blow up at $r=0$. Thus,

$$V(r,\theta)=\sum_{m=0}^{\infty}A_m r^m P_m(\cos\theta)\ .$$

The potential at $r=R$ is therefore equal to

$$V(r,\theta)=\sum_{m=0}^{\infty}A_m R^m P_m(\cos\theta)=k(\cos 3\theta)\ .$$

Using the trigonometric relations we can rewrite $\cos(3\theta)$ as

$$\cos(3\theta)=4\cos^3\theta-3\cos\theta=\frac{8}{5}P_3(\cos\theta)-\frac{3}{5}P_1(\cos\theta)\ .$$

Substituting this expression in the equation for $V(r,\theta)$ we obtain

$$V(R,\theta)=\sum_{m=0}^{\infty}A_m R^m P_m(\cos\theta)=\frac{8}{5}kP_3(\cos\theta)-\frac{3}{5}kP_1(\cos\theta)\ .$$

This equation immediately shows that $A_m=0$ unless $m=1$ or $m=3$ then

$$A_1=\frac{3}{5}\frac{k}{R}\ ,\qquad A_3=\frac{8}{5}\frac{k}{R^3}\ .$$

Therefore, the electrostatic potential inside the sphere is equal to

$$V(r,\theta) = \frac{8}{5}k\frac{r^3}{R^3}P_3(\cos\theta) - \frac{3}{5}k\frac{r}{R}P_1(\cos\theta)$$

Now, one can show that the solution of the electrostatic potential outside the sphere $(r > R)$ is thus equal to

$$V(r,\theta) = \frac{-3}{5}k\frac{R^2}{r^2}P_1(\cos\theta) + \frac{8}{5}k\frac{R^4}{r^4}P_3(\cos\theta)\ .$$

3.6 The Method of Eigen-Function Expansion

The separation of variables technique requires the boundary value problem (BVP) to have a homogeneous partial differential equation (PDE) and homogeneous boundary conditions (BCs). The initial conditions are not generally homogeneous and they do not prevent the BVP from being solved. Here we will solve the case of non-homogeneous boundary conditions [4]. To highlight the power of Eigen-function expansion method, we will tackle a Poisson boundary value problem on a rectangular region, as described by figure(3.1) below, involving Poisson's equation.

3.6.1 Eigen-Function Expansion Method for Poisson's Equation

Poisson's Equation is an elliptic linear inhomogeneous partial differential equation of the second order. It is given by

$$\nabla^2 u = -f$$

or
$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f\ (x, y) \tag{3.24}$$

The solution of the equation u, is the unknown scalar potential function. When the Poisson equation is satisfied by a scalar potential in a given domain Ω. We can find the scalar potential inside the domain by solving the Poisson equation with the help of boundary conditions prescribed at the boundary $\partial\Omega$. Poisson equation arises in a variety of branches mathematical and physical It is used to describe the steady state conditions of a given system as it involves no time variable.

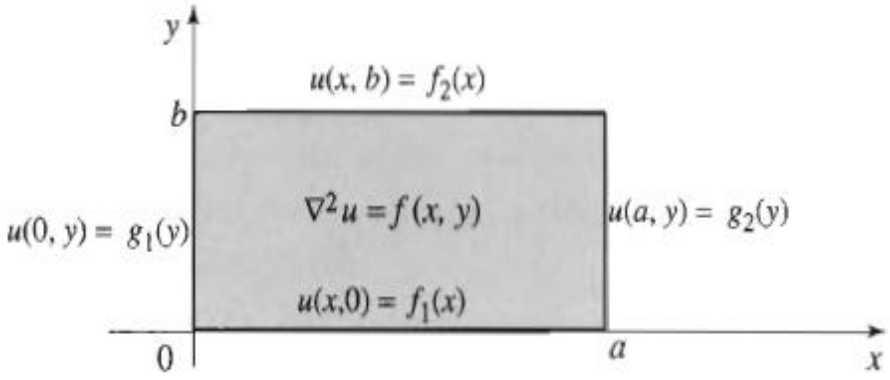
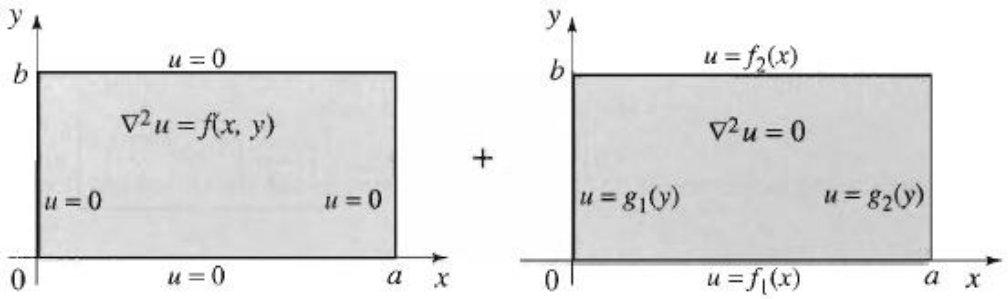


Figure (3.1): A general Poisson problem of a rectangle

The first step is decomposing the problem into simpler sub-problems, the decomposition is described by figure (3.2)



(a) Poisson problem with zero boundary data.

(b) A general Dirichlet problem.

Figure (3.2): Decomposition of a general Poisson problem

Figure (3.2.b) describes a Dirichlet problem on a rectangle when f_1, g_1,and f_2, g_2 are all equal zeros, by solving a boundary value problem using the method of separation of variables, we will arrive with the general form of the solution

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{a}x \sinh\frac{n\pi}{a}y \ .$$

and $$B_n = \frac{2}{a\sinh\frac{n\pi b}{a}} \int_0^a f_2(x)\sin\frac{n\pi}{a}x \, dx \,, \quad n = 1,2,...$$

if we consider $\phi_{mn}(x,y)$ to be $= \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y$ which clearly satisfies the zero boundary conditions in figure(3.2.a). computing the Laplacian of $\phi_{mn}(x,y)$ we find

$$\nabla^2\phi(x,y)=\sin\frac{n\pi}{b}y\,\frac{\partial^2}{\partial x^2}(\sin\frac{m\pi}{a}x)+\sin\frac{m\pi}{a}x\,\frac{\partial^2}{\partial y^2}(\sin\frac{n\pi}{b}y)$$

$$=-\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]\sin\frac{m\pi}{a}x\ \sin\frac{n\pi}{b}y\ .$$

but we call the constant $\lambda_{mn}=\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2$ (m,n=1,2,...)

an eigenvalue of the Laplacian and $\phi_{mn}(x,y)=\sin\frac{m\pi}{a}x\ \sin\frac{n\pi}{b}y$

the corresponding eigenfunction. So we arrive with the solution

$$u(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}E_{mn}\sin\frac{m\pi}{a}x\ \sin\frac{n\pi}{b}y\ ,$$

Differentiating twice and plugging into (3.24) gives

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}-E_{mn}\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]\sin\frac{m\pi}{a}x\ \sin\frac{n\pi}{b}y\ =f\ (x,y)$$

This is a double Fourier sine series expansion of $f\ (x,y)$, solving for E_{mn}

we get

$$E_{mn}=\frac{-4}{ab\lambda_{mn}}\int_0^b\int_0^a f\ (x,y)\sin\frac{m\pi}{a}x\ \ \sin\frac{n\pi}{b}y\ dx\ dy\,.$$

3.6.2 Eigen-Function Expansion Method for The IBVP

Now suppose we have the non-homogeneous wave equation [24]

$$u_{tt} - c^2 u_{xx} = S(x,t)$$

Subject to the conditions

$$u(x,0) = f(x)\ ,\ u_t(x,0) = 0$$

$$u(0,t) = u(l,t) = 0$$

Solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\,\phi_n(x)$$

$$S(x,t) = \sum_{n=1}^{\infty} s_n(t)\,\phi_n(x)$$

$$\left.\begin{array}{c} \phi_n(x) = \sin\dfrac{n\pi}{L}x \\ \lambda_n = \left(\dfrac{n\pi}{L}\right)^2 \end{array}\right\} \quad n = 1,2,\ldots$$

$$s_n(t) = \frac{\int_0^L S(x,t)\sin\frac{n\pi}{L}x\,dx}{\int_0^L \sin^2\frac{n\pi}{L}x\,dx}$$

$$u_n''(t) + c^2\left(\frac{n\pi}{L}\right)^2 u_n(t) = s_n(t)$$

$$u_n(t) = c_1 \cos\frac{n\pi}{L} ct + c_2 \sin\frac{n\pi}{L} ct + \int_0^t s_n(\tau) \frac{\operatorname{sinc}\frac{n\pi}{L}(t-\tau)}{c\frac{n\pi}{L}} d\tau$$

$$u_n(0) = c_1 = \frac{\int_0^\infty f(x) \sin\frac{n\pi}{L} x\, dx}{\int_0^\infty \sin^2\frac{n\pi}{L} x\, dx} \quad \text{since } u(x,0) = f(x)$$

$$u_n'(0) = c_2\, c\frac{n\pi}{L} = 0 \quad \text{since} \quad u_t(x,0) = 0 \;\Rightarrow\; c_2 = 0$$

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ c_1 \cos\frac{n\pi}{L} ct + \frac{L}{cn\pi} \int_0^t s_n(\tau) \operatorname{sinc}\frac{n\pi}{L}(t-\tau)\, d\tau \right\} \sin\frac{n\pi}{L} x$$

Where c_1 is given above.

3.7 Integral Transforms

3.7.1 Laplace Transform

Let $u(x,t)$ be an arbitrary function defined for $a \le x \le b$ and $t > 0$, then the Laplace transform of $u(x,t)$ with respect to t is defined as

$$\mathcal{L}[u(x,t)] = \tilde{U}(x,s) = \int_0^\infty u(x,t) e^{-st} dt$$

In order to use Laplace transforms in practice, a formula is required to transform back from $\tilde{U}(x,s)$ to $u(x,t)$ that is;

$$\mathcal{L}^{-1}[\tilde{U}(x,s)] = u(x,t) = \int_0^\infty \tilde{U}(x,s) e^{-st} ds$$

As the range of integration for Laplace transforms is infinite [9], only those independent variables for which the dependent variable of the partial differential equation is defined over an infinite range are suitable as

variables of transformation. Laplace transforms are Particularly suited to problems where boundary conditions are given at $t = 0$ Such problems arise in the solution of the heat equation and wave equation, the independent variable t being interpreted there as the time variable. Here attention is confined to transforming the time variable t .

Also we have the Laplace's transform for the first derivative as:

$$\mathcal{L}\left[\frac{\partial u}{\partial t}\right] = \int_0^\infty \frac{\partial u}{\partial t} e^{-st} dt = u(x,t)e^{-st}\Big|_0^\infty + s\int_0^\infty u(x,t)e^{-st}dt$$

$$\mathcal{L}\left[\frac{\partial u}{\partial t}\right] = s\tilde{U}(x,s) - u(x,0)$$

In a similar fashion we have Laplace's transform for the second derivative as: $\mathcal{L}\left[\frac{\partial^2 u}{\partial t^2}\right] = \int_0^\infty \frac{\partial^2 u}{\partial t^2} e^{-st} dt = \frac{\partial u(x,t)}{\partial t} e^{-st}\Big|_0^\infty + s\int_0^\infty \frac{\partial u}{\partial t} e^{-st} dt$

or

$$\mathcal{L}\left[\frac{\partial^2 u}{\partial t^2}\right] = s\mathcal{L}\left[\frac{\partial u}{\partial t}\right] - u_t(x,0)$$

$$= s^2\tilde{U}(x,s) - s\,u(x,0) - u_t(x,0).$$

Also, in similar fashion; we can obtain Laplace's transform for the first derivative and the second derivative respectively, as:

$$\mathcal{L}\left[\frac{\partial u}{\partial t}\right] = \frac{\mathrm{d}\tilde{U}(x,s)}{\mathrm{d}x}$$

$$\mathcal{L}\left[\frac{\partial^2 u}{\partial t^2}\right] = \frac{\mathrm{d}^2\tilde{U}(x,s)}{\mathrm{d}x^2}$$

Example (3.1)

Consider this one dimensional wave problem and we want to solve it by Laplace Transform;

$$\frac{\partial^2 u}{\partial x^2}(x,t) = 4\frac{\partial^2 u}{\partial t^2}(x,t)\ ,\ \ t>0\ ,\ \ x>0\ ,$$

subject to the conditions:

$$u(x,0)=0\ ,\ u_t(x,0)=-1\ ,\ x>0.$$

$$u(0,t)=t\ \ ,\ \ x>0.$$

$\lim_{x\to\infty} u(x,t)$ exists for a fixed $t>0$.

Solution:

Taking Laplace transform relative to t since x has condition at $(x=0)$, gives

$$\mathcal{L}\{u(x,t)\} = \tilde{U}(x,s) = \int_0^\infty u(x,t)\,e^{-st}dt.$$

$$\mathcal{L}\{u_{xx}\} = \frac{\partial^2 \mathcal{L}\{u\}}{\partial x^2} = 4\mathcal{L}\{u_{tt}\}$$

$$\frac{\partial^2 \tilde{U}}{\partial x^2} = 4(s^2\tilde{U}+1) \quad \text{or} \quad \frac{\partial^2 \tilde{U}}{\partial x^2} - 4s^2\tilde{U} = 4$$

This ODE has the solution

$$\tilde{U}(x,s) = c_1(s)\,e^{-2sx} + c_2\,e^{2sx} - \frac{1}{s^2}\ .$$

But, $\lim_{x\to\infty} u(x,t)$ exists, so we can

write $\int_0^\infty e^{-st}[\lim_{x\to\infty} u(x,t)\,]dt = \lim_{x\to\infty}\int_0^\infty e^{-st}u(x,t)\,dt = \lim_{x\to\infty}\tilde{U}(x,s)$

Thus, we obtain $c_2(s)=0$,

thus $$\tilde{U}(x,s)=c_1(s)\,e^{-2sx}-\frac{1}{s^2}$$

since we have $u(0,t)=t$, then taking Laplace transform to $u(0,t)=t$, we obtain

$$\mathcal{L}\{u(0,t)\}=\tilde{U}(0,s)=\int_0^\infty t e^{-st}dt=\frac{1}{s^2}$$

Substituting this for $\tilde{U}(x,s)$, we have

$$\tilde{U}(0,s)=c_1(s)-\frac{1}{s^2}=\frac{1}{s^2}\ ,\quad \text{means that}\quad c_1(s)=\frac{2}{s^2}.$$

so $$\tilde{U}(x,s)=\frac{2}{s^2}e^{-2sx}-\frac{1}{s^2}$$

The solution to the original problem is by taking the inverse of Laplace transform,

$$u(x,t)=-t+2(t-2x)\,u(t-2x).$$

3.7.2 The Fourier Transform Method

Maxwell's equations are a system of eight first-order partial differential equations in four independent variables: three space coordinates and time,

whose solution is often quite complicated [4,9]. It may be advantageous to eliminate the dependence of the field quantities upon one or more of the independent variables by applying a Fourier (or Laplace) transform , solving the resulting equations in the transform domain, and then obtaining the desired field quantities by an inverse transformation. Obviously, the main advantage of a transform technique with respect to an independent variable is to change the dependence of the equations on that variable from a differential one to an algebraic one; thus, a four-fold Fourier transform can change the differential system to an algebraic system in the transform domain.

The Fourier transform pair:

$$\boldsymbol{E}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} \boldsymbol{E}(\mathbf{r}, t)\, e^{-j\omega t} dt,$$

$$\boldsymbol{E}(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(\mathbf{r}, \omega)\, e^{j\omega t} d\omega,$$

allows us to transform the electric field from the time domain, where the appropriate field vector is $\boldsymbol{E}(\mathbf{r}, t)$, to the frequency domain, where the appropriate field vector is $\boldsymbol{E}(\mathbf{r}, \omega)$, and vice versa. Identical transformations can be applied to all field variables in Maxwell's equations system.

When modeling problems over regions that extend very far in at least one direction, we can often idealize the situation to that of a problem having infinite extent in one or more directions. Here we will develop the Fourier

transform method and apply it to solve the wave equations on the real line. The appropriate tools for solving a problems on the semi-infinite interval include the cosine and sine of Fourier transforms and the Laplace transforms. Here we will suppose that $u(x,t)$ is a function of two variables x and t, where $-\infty < x < \infty$ and $t > 0$. For fixed t, the function $u(x,t)$ becomes a function of the spatial variable x, and so we can take its Fourier transform with respect to the x variable [9]. We denote the transform by $\tilde{U}(\omega,t)$, thus

$$F(u(x,t))(\omega) = \tilde{U}(\omega,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{-j\omega x}dx \ . \tag{3.25}$$

Fourier Transform and Partial Derivatives

$$F(\frac{\partial}{\partial t}u(x,t))(\omega) = \frac{d}{dt}\tilde{U}(\omega,t)\ ; \tag{3.26}$$

$$F(\frac{\partial^n}{\partial t^n}u(x,t))(\omega) = \frac{d^n}{dt^n}\tilde{U}(\omega,t)\ ;\ \text{n=1,2,...}\ . \tag{3.27}$$

$$F(\frac{\partial}{\partial x}u(x,t))(\omega) = j\omega\tilde{U}(\omega,t)\ ; \tag{3.28}$$

$$F(\frac{\partial^n}{\partial x^n}u(x,t))(\omega) = (j\omega)^n\tilde{U}(\omega,t)\ ;\ \text{n=1,2,...}\ . \tag{3.29}$$

Example (3.2)

Let us consider the one dimensional wave equation with its initial conditions as follows:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad (-\infty < x < \infty, \ t > 0),$$

with the initial conditions

$$u(x, 0) = f(x)$$

$$\frac{\partial}{\partial t} u(x, 0) = g(x)$$

Solution :

We take the Fourier transform of both sides of this partial differential equation and the initial conditions with respect to x , using Eq.(3.27) and Eq.(3.29) with n=2, we get

$$\frac{d^2}{dt^2} \tilde{U}(\omega, t) = -c^2 \omega^2 \tilde{U}(\omega, t) \tag{3.30}$$

$$\tilde{U}(\omega, 0) = \tilde{f}(\omega), \tag{3.31}$$

$$\frac{d}{dt} \tilde{U}(\omega, 0) = \tilde{g}(\omega). \tag{3.32}$$

It is clear that equation (3.30) is an ordinary differential equation in $\tilde{U}(\omega, t)$, where t is the variable. If we write (3.30) in the standard form

$$\frac{d^2}{dt^2} \tilde{U}(\omega, t) + c^2 \omega^2 \tilde{U}(\omega, t) = 0$$

The general solution of this equation is

$$\tilde{U}(\omega, t) = A(\omega) \cos c\omega t + B(\omega) \sin c\omega t \ ,$$

Where $A(\omega)$ and $B(\omega)$ are constants in t , we determine $A(\omega)$ and $B(\omega)$ from the initial conditions (3.31) and (3.32) as follows:

$$\tilde{U}(\omega, 0) = A(\omega) = \tilde{f}(\omega) ,$$

$$\frac{d}{dt}\tilde{U}(\omega, 0) = c\omega B(\omega) = \tilde{g}(\omega) .$$

so

$$\tilde{U}(\omega, t) = \tilde{f}(\omega)\cos c\omega t + \frac{1}{c\omega}\tilde{g}(\omega)\sin c\omega t .$$

To obtain the solution we use the inverse Fourier transform and get

$$u(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\tilde{f}(\omega)\cos c\omega t + \frac{1}{c\omega}\tilde{g}(\omega)\sin c\omega t]e^{j\omega x} d\omega .$$

3.7.3 The Fourier Sine and Cosine Transform

Without proofs, the Fourier cosine and sine integral representation are given [28] respectively :

The Fourier cosine Transform of u is

$$\tilde{U}_c(\omega, t) = \int_0^{\infty} u(x, t)\cos \omega x \, dx \quad (\omega \geq 0) .$$

With an inverse transform

$$u(x, t) = \frac{2}{\pi}\int_0^{\infty} \tilde{U}_c(\omega, t)\cos \omega x \, d\omega \quad (x > 0)$$

in similar way, we define the Fourier sine Transform of u by

$$\tilde{U}_s(\omega,t)=\int_0^\infty u(x,t)\,\sin\omega x\,dx \quad (\omega\geq 0)\ .$$

With an inverse Fourier sine transform

$$u(x,t)=\frac{2}{\pi}\int_0^\infty \tilde{U}_s(\omega,t)\sin\omega x\,d\omega \quad (\mathrm{x}>0)\ .$$

The sine transform of $\frac{\partial u}{\partial x}$ with respect to x is given by

$$\int_0^\infty \frac{\partial u}{\partial x}\sin\omega x\,dx=[u\sin\omega x]_0^\infty-\int_0^\infty u(x,t)\cos\omega x\,dx\ .$$

Provided $u(x,t)\to 0 \ \ as\ \ x\to\infty$ (which is often the case in physical problems), then

$$\int_0^\infty \frac{\partial u}{\partial x}\sin\omega x\,dx=-\omega\,\tilde{U}_c(\omega,t)\ .$$

Similarly the cosine transform of $\frac{\partial u}{\partial x}$ with respect to x is

$$\int_0^\infty \frac{\partial u}{\partial x}\cos\omega x\,dx=[u\cos\omega x]_0^\infty+\omega\int_0^\infty u(x,t)\sin\omega x\,dx\ .$$

$$=\omega\tilde{U}_s(\omega,t)-u(0,t),$$

provided $u(x,t)\to 0 \ \ as\ \ x\to\infty$.

The transform of the second derivative $\frac{\partial^2 u}{\partial x^2}$ may be obtained in a like fashion, and we easily can find

$$\int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin\omega x \, dx = -\omega^2 \tilde{U}_s(\omega, t) + \omega u(0, t),$$

and

$$\int_0^\infty \frac{\partial^2 u}{\partial x^2} \cos\omega x \, dx = -\omega^2 \tilde{U}_c(\omega, t) - \left(\frac{\partial u}{\partial x}\right)_{x=0}$$

where, in addition to assuming that $u(x, t) \to 0 \quad as \quad x \to \infty$, we have further assumed that $\frac{\partial u}{\partial x} \to 0 \quad as \quad x \to \infty$.

other commonly used notation is the following

$$F_c(f) = \tilde{f}_c \quad \text{and} \quad F_s(f) = \tilde{f}_s .$$

Example (3.3)

A Dirichlet-Neumann Problem in a Semi-Infinite Strip

Consider the Dirichlet- Neumann problem of an infinitely long rectangular conducting strip which cross section is shown in figure (3.3). This is a boundary value problem [4], and the PDE to be solved is :

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad y > 0, \tag{3.33}$$

$$\frac{\partial u}{\partial y}(x,0) = 0, \quad 0 < x < a, \tag{3.34}$$

$$u(0,y) = 0, \quad u(a,y) = f(y), \; y > 0. \tag{3.35}$$

As illustrated in the figure below :

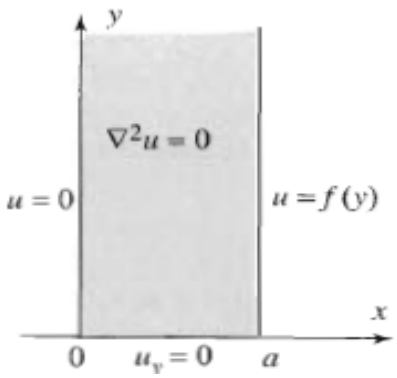


Figure (3.3): Dirichlet-Neumann problem in a semi infinite strip

Solution:

Since the domain of the variable y is semi-infinite (the domain of the variable x is finite) we choose to transform the equations with respect to the variable y . Also, since the boundary condition (3.34) involves the derivative at $y = 0$, the cosine transform is the right choice. So we obtain

$$F_c\left(\frac{\partial^2 u}{\partial y^2}\right) = -\omega^2 \tilde{U}_c(x,\omega) - \frac{\partial u}{\partial y}(x,0) = -\omega^2 \tilde{U}_c(x,\omega),$$

$$\frac{d^2}{dx^2}\tilde{U}_c(x,\omega) - \omega^2 \tilde{U}_c(x,\omega) = 0,$$

$$\tilde{U}_c(0,\omega) = 0,\ \tilde{U}_c(a,\omega) = \tilde{f}_c(\omega),$$

The general solution of the second order ordinary differential equation is $\tilde{U}_c(x,\omega) = A(\omega)\cosh \omega x + B(\omega)\sinh \omega x$,

Where $A(\omega)$ and $B(\omega)$ are constants that depend on ω. Setting $x = 0$ and then $x = a$, we get

$$A(\omega) = 0, \quad B(\omega) = \frac{\tilde{f}_c(\omega)}{\sinh \omega a}.$$

Putting this into $\tilde{U}_c(x, \omega)$ and taking inverse Fourier cosine transform , we get the solution in the form

$$u(x,y) = \int_0^\infty \tilde{U}_c(x, \omega)\cos \omega y \, d\omega = \int_0^\infty \frac{\tilde{f}_c(\omega)}{\sinh \omega a}\sinh \omega x \cos \omega y \, dy \ .$$

Example (3.4)

Suppose we want to determine the potential function for the region inside the rectangular trough of infinite length , this is a two dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

in the half plane $y \geq 0$ subject to the boundary condition $u(x,0) = f(x)$ $(-\infty < x < \infty)$ and the condition $u(x,t) \to 0 \quad as \quad \sqrt{x^2 + y^2} \to \infty.$

Solution:

Using a Fourier transform [9] with respect to x

$$F(u(x,\mathrm{y})) = \tilde{U}(\omega,\mathrm{y}) = \int_{-\infty}^{\infty} u(x,\mathrm{y})\, e^{j\omega x} dx \ .$$

and $$F\left(\frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial^2 \tilde{U}}{\partial y^2}, \quad F\left(\frac{\partial^2 u}{\partial x^2}\right) = (-j\omega)^2 \tilde{U}.$$

which implies

$$\frac{\partial^2 \tilde{U}}{\partial y^2} - \omega^2 \tilde{U} = 0.$$

with the solution

$$\tilde{U}(\omega, y) = A e^{\omega y} + B e^{-\omega y}.$$

The boundary conditions give

$$\tilde{U}(\omega, 0) = F(f) = F(\omega)$$

and $\tilde{U}(\omega,\mathrm{y}) \to 0 \ as \ y \to \infty$

if $\omega > 0,$ we must have $A = 0, \ B = F(\omega),$

$\omega < 0,$ we must have $B = 0, \ A = F(\omega),$

which gives

$$\tilde{U}(\omega, y) = F(\omega) e^{-|\omega| y}$$

Now taking the inverse gives

$$F^{-1}(e^{-|\omega|y}) = \frac{1}{\pi}\frac{y}{x^2+y^2}$$

so the convolution theorem yields

$$u(x,y) = \int_{-\infty}^{\infty} f(x-u)\frac{1}{\pi}\frac{y}{u^2+y^2}du$$

$$= \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(x-u)}{u^2+y^2}du$$

$$= \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(u)}{(u-x)^2+y^2}du \ .$$

3.8 Green's Function

Physically, Green's functions associated with the wave type problems, and represents the way in which a wave propagates from one point in space to another. For this reason, they are sometimes referred to as propagators [9]. In this case, the Green's function is a function of the "path length" between x and x_0, irrespective of whether $x > x_0$ or $x < x_0$. The path length is given by $|x - x_0|$, and the Green's function is a function of this path length which is why, using the notation $x|x_0 \equiv |x - x_0|$,we write $g(x|x_0)$.

3.8.1 Green's Function for The One-Dimensional Inhomogeneous Wave Equation.

Consider the equation

$$(\frac{\partial^2}{\partial x^2} + k^2)\, u(x,k) = f(x) \tag{3.36}$$

where k (the wave number) is a constant and $f(x)$ is the source term, the solution being required over all space $x \in (-\infty,\infty)$ subject to the conditions that u and $\partial u/\partial x$ are zero at $\pm\infty$. This equation describes the behavior of "steady waves" (constant wavelength $\lambda = 2\pi / k$) due to a source $f(x)$.

We define the Green's function as being the solution to the equation obtained by replacing the source term with a delta function which represents a point source at x_0, giving the equation

$$(\frac{\partial^2}{\partial x^2} + k^2)\, g(x|x_0,k) = \delta(x - x_0) \tag{3.37}$$

where δ has the following fundamental property

$$\int_{-\infty}^{\infty} u(x)\, \delta(x - x_0)\, dx = u(x_0).$$

Multiplying equation (3.36) by g gives

$$g\,(\frac{\partial^2}{\partial x^2} + k^2)\, u = gf\ ,$$

and multiplying equation (3.37) by u gives

$$u\,(\frac{\partial^2}{\partial x^2}+k^2)\,g = u\;\delta(x-x_0). \tag{3.38}$$

Now subtract the two results and integrate to obtain

$$\int_{-\infty}^{\infty}(g\,\frac{\partial^2 u}{\partial x^2}-u\,\frac{\partial^2 g}{\partial x^2})\,dx = \int_{-\infty}^{\infty} fg\;dx - \int_{-\infty}^{\infty} u\;\delta(x-x_0)\,dx\;.$$

Using the generalized sampling property of the delta function given above and evaluating the second integral on the right-hand side and using the conditions above, we obtain the Green's function solution to equation (3.36) in the form

$$u(x_0,k) = \int_{-\infty}^{\infty} f\;(x)\,g(x|x_0,k)\,dx\;.$$

where g is the Green's function. This solution is of course worthless without an expression for the Green's function which is given by the solution to the equation

$$(\frac{\partial^2}{\partial x^2}+k^2)\,g(x|x_0,k) = \delta(x-x_0)$$

subject to, $g(x|x_0,k)\big|_{\pm\infty} = 0$, and $[\partial g(x|x_0,k)/\partial x]_{\pm\infty} = 0$.

The solution to this equation is based on employing the properties of the Fourier transform discussed above. Writing $X = |x-x_0|$, we express g and δ as Fourier transforms, that is

$$g(X,k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(u,k)\,e^{iuX}\,du \tag{3.39}$$

and $$\delta(X) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iuX}\,du$$

Substituting these expressions into equation (3.38) and differentiating gives

$$G(u,k) = \frac{1}{u^2 - k^2} \ .$$

Substituting this result back into equation (3.39) we obtain

$$g(X,k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{iuX}}{u^2 - k^2}\,du = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{iuX}}{(u-k)(u+k)}\,du \ . \qquad (3.40)$$

3.8.2 Green's Function Solution to Maxwell's Equations Time-Dependent Problems

Here we will introduce the Green's function as a tool for predicting the behavior of electromagnetic fields. We start by considering the basic equations of electromagnetism (Maxwell's equation) and show how these can be manipulated (under certain conditions) to form inhomogeneous wave equations for related electromagnetic field potentials [9]. The Green's function is then used to solve these equations which leads directly to a description of the properties of an electromagnetic field. The previous discussions having been related to the time-independent case.

The motion of electrons (and other charged particles) gives rise to electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ fields. These fields are related by the (microscopic) Maxwell's equations which are as follows:

Coulomb's law

$$\nabla \cdot \mathbf{E} = 4\pi\rho \ . \tag{3.41}$$

where ρ is the charge density.

Faraday's law

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \ . \tag{3.42}$$

where c is the speech of light (in a vacuum).

No free magnetic poles law

$$\nabla \cdot \mathbf{B} = 0 \ . \tag{3.43}$$

Modified (by Maxwell) Ampere's law

$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J} \ . \tag{3.44}$$

where $\mathbf{J}$ is the current density. These microscopic Maxwell's equations are used to predict the point wise electric and magnetic fields given the charge and current densities (ρ and $\mathbf{J}$ respectively).

3.8.3 The Wave Equation Solution of Maxwell's Equations

If we take the curl of equation (3.42)

$$\nabla \times \nabla \times \mathbf{E} = -\frac{1}{c} \nabla \times \frac{\partial \mathbf{B}}{\partial t} \ ,$$

and use the vector identity

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \ , \qquad (*)$$

then from equations (3.41) and (3.42), we obtain

$$\nabla (4\pi\rho) - \nabla^2 \mathbf{E} = -\frac{1}{c} \frac{\partial}{\partial t} \left(\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \mathbf{J} \right).$$

After rearranging,

$$\nabla^2 \mathbf{E} - \frac{1}{c} \frac{\partial^2 \mathbf{E}}{\partial t} = 4\pi \nabla \rho + \frac{4\pi}{c^2} \frac{\partial \mathbf{J}}{\partial t}. \qquad (3.45)$$

Taking the curl of equation (3.44), using the same vector identity as above (*), substituting for equations (3.42) and (3.43) and rearranging the result gives

$$\nabla^2\mathbf{B}-\frac{1}{c}\frac{\partial^2\mathbf{B}}{\partial t}=-\frac{4\pi}{c^2}\nabla\times\mathbf{J}. \qquad (3.46)$$

Equations (3.45) and (3.46) are the inhomogeneous wave equations for $\mathbf{E}$ and $\mathbf{B}$ .They are related or coupled with the vector field $\mathbf{J}$ (which is related to $\mathbf{B}$). If we define a region of free space where $\rho=0$ and $\mathbf{J}=0$, then both $\mathbf{E}$ and $\mathbf{B}$ satisfy

$$\nabla^2\mathbf{F}-\frac{1}{c^2}\frac{\partial^2\mathbf{F}}{\partial t^2}=0\ ,$$

which is the homogeneous wave equation prescribed above.

3.8.4 The General Solution of Maxwell's Equations

The basic method of solving Maxwell's equations (i.e. finding $\mathbf{E}$ and $\mathbf{B}$ given ρ and $\mathbf{J}$) [9] involves the following :

- Expressing $\mathbf{E}$ and $\mathbf{B}$ in terms of two other fields U and $\mathbf{A}$.
- Obtaining two separate equations for U and $\mathbf{A}$.
- Solving these equations for U and $\mathbf{A}$ from which $\mathbf{E}$ and $\mathbf{B}$
- can then be computed.
-

For any vector field $\mathbf{A}$

$$\nabla.\nabla\times\mathbf{A}=0$$

Hence, if we write

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{3.47}$$

Equation (3.43) remains unchanged, and equation (3.42) can then be written as

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\nabla \times \mathbf{A},$$

or

$$\nabla \times (\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}) = 0.$$

The field $\mathbf{A}$ is called the magnetic vector potential. Similarly, for any scalar field U

$$\nabla \times \nabla U = 0,$$

and thus equation (3.42) is satisfied if we write

$$\pm \nabla U = \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

or

$$\mathbf{E} = -\nabla U - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \tag{3.48}$$

The field U is called the electric scalar potential. Substituting equation (3.48) into Maxwell's equation (3.41) gives

$$\nabla.(\nabla U + \frac{1}{c}\frac{\partial A}{\partial t}) = -4\pi\rho,$$

or

$$\nabla^2 U + \frac{1}{c}\frac{\partial}{\partial t}\nabla.\mathbf{A} = -4\pi\rho \ . \tag{3.49}$$

Substituting equations (3.47) and (3.48) into Maxwell's equation (3.44) gives

$$\nabla\times\nabla\times\mathbf{A} + \frac{1}{c}\frac{\partial}{\partial t}(\nabla U + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}) = \frac{4\pi}{c}\mathbf{J} \ .$$

Using the identity

$$\nabla\times\nabla\times\mathbf{A} = \nabla\nabla.\mathbf{A} - \nabla^2\mathbf{A}$$

This becomes

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla(\nabla.\mathbf{A} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}) = -\frac{4\pi}{c}\mathbf{J} \ . \tag{3.50}$$

If we could solve equations (3.49) and (3.50) for U and $\mathbf{A}$ then $\mathbf{E}$ and $\mathbf{B}$ could be computed. However, these equations are coupled. They can be decoupled if we introduce a condition known as the Lorentz condition:

$$\nabla . \mathbf{A} + \frac{1}{c}\frac{\partial U}{\partial t} = 0 \ . \tag{3.51}$$

Substituting equation (3.51) into equations (3.49) and (3.50) gives

$$\nabla^2 U - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = -4\pi\rho \ ,$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\rho \ .$$

respectively. These equations are uncoupled inhomogeneous wave equations. Unlike the wave equations that we have considered before, these equations are time dependent and it is therefore pertinent at this point to consider the Green's function for a time-dependent wave equation.

3.8.5 Green's Functions for Time-Dependent Inhomogeneous Wave Equations

First we will consider three-dimensional problem but stress that the methods of solution discussed here can be applied directly to problems in one and two dimensions [9]. Thus, consider the case in which a time varying source function $f(\mathbf{r}, t)$ produces a wave field U which is taken to be the solution to the equation**.**

$$\nabla^2 U - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} U(\mathbf{r}, t) = -f(\mathbf{r}, t) \ . \tag{3.52}$$

As with the time-independent problem, the Green's function for this equation is defined as the solution to the equation obtained by replacing $f(\mathbf{r},t)$ with $\delta^3(\mathbf{r}-\mathbf{r}_0)\delta(t-t_0)$ that is the solution to the equation

$$(\nabla^2+\frac{1}{c^2}\frac{\partial^2}{\partial t^2})\,G(\mathbf{r}|\mathbf{r}_0,t|t_0)=\delta^3(\mathbf{r}-\mathbf{r}_0)\delta(t-t_0)\ . \qquad (3.53)$$

where G is used to denote the time-dependent Green's function, $\mathbf{r}_0$ is the position of the source and $t|t_0=t-t_0$. To obtain the equation for the time-independent Green's function, we write G and $\delta(t-t_0)$ as Fourier transforms,

$$G(\mathbf{r}|\mathbf{r}_0,t|t_0)=\frac{1}{2\pi}\int_{-\infty}^{\infty} g(\mathbf{r}|\mathbf{r}_0,\omega)\,e^{[i\omega(t-t_0)]}d\omega$$

and

$$\delta(t-t_0)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{[i\omega(t-t_0)]}d\omega$$

where ω is the angular frequency. Substituting these equations into equation (3.53) we then obtain

$$(\nabla^2+k^2)g(\mathbf{r}|\mathbf{r}_0,k)=-\delta^3(\mathbf{r}-\mathbf{r}_0)$$

which is the same equation as that used previously to define the time-independent Green's function. Thus, once g has been obtained, the time-dependent Green's function can be derived by computing the Fourier integral given above. Using the expression for g derived earlier in section (3.9.1),

$$G(\mathbf{r}|\mathbf{r}_0, t|t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{4\pi|r-r_0|}\exp(ik|r-r_0|)\exp[i\omega(t-t_0)]\,d\omega$$
$$= \frac{1}{4\pi|r-r_0|}\delta(t-t_0+|r-r_0|/c).$$

3.9 Conformal Mapping Method

Conformal mapping is an important technique used in complex analysis and has many applications in different physical situations. If the function is harmonic (i.e it satisfies Laplace's equation $\nabla^2 f = 0$) then the transformation of such functions via conformal mapping is also harmonic. A large number of problems arising in fluid mechanics, electrostatics, heat conduction, and many other physical situations can be mathematically formulated in terms of Laplace's equation, i.e, all these physical problems reduce to solving the equation

$$\Phi_{xx} + \Phi_{yy} = 0 \tag{3.54}$$

in a certain region D of the z plane. The function $\Phi(x, y)$, in addition to satisfying this equation also satisfies certain boundary conditions on the boundary C of the region D [18]. From the theory of analytic functions we know that the real and the imaginary parts of an analytic function satisfy Laplace's equation. It follows that solving the above problem reduces to finding a function that is analytic in D and that satisfies certain boundary conditions on C. It turns out that the solution of this problem can be greatly simplified if the region D is either the upper half of the z plane or the unit disk. What makes conformal mapping so useful is that they map a solution

of Laplace's equation into another solution of Laplace's equation.The driving force behind many of the applications of complex analysis is the remarkable connection between harmonic functions of two variables. solutions of the planar Laplace equation and complex functions. To wit, the real and imaginary parts of any complex analytic function are automatically harmonic. We interpret this operation as a complex changes of variables ,also known as a conformal mapping since it preserves angles [18]. There are several ways to motivate the link between harmonic functions $u(x, y)$, meaning solutions of the two-dimensional Laplace equation in equation (3.54), and complex functions $f(z)$. One natural starting point is to return to the D'Alembert solution of the one-dimensional wave equation, which was based on the factorization

$$\partial_t^2 - c^2 \partial_x^2 = (\partial_t - c\partial_x)(\partial_t + c\partial_x)$$

of the linear wave operator. The two-dimensional Laplace operator $\Delta = \partial_x^2 + \partial_y^2$ has essentially the same form, except for a "minor" change in sign. The Laplace operator admits a complex factorization,

$$\Delta = \partial_x^2 + \partial_y^2 = (\partial_x - \mathrm{i}\partial_y)(\partial_x + \mathrm{i}\partial_y) ,$$

into a product of first order differential operators, with complex "wave speeds" $c = \pm \mathrm{i}$. The solutions to the Laplace equation (3.54) should be expressed in the form

$$u(x,y) = f(x + \mathrm{i}y) + g(x - \mathrm{i}y)\,, \tag{3.55}$$

i.e., a linear combination of functions of the complex variable $z = x + \mathrm{i}y$ and its complex conjugate $z = x - \mathrm{i}y$. The functions $f(x + \mathrm{i}y)$ and $g(x - \mathrm{i}y)$ formally satisfy the first order complex partial differential equations

$$\frac{\partial f}{\partial x} = \mathrm{i}\frac{\partial f}{\partial y}\,, \qquad \frac{\partial g}{\partial x} = -\mathrm{i}\frac{\partial g}{\partial y}\,, \tag{3.56}$$

In most applications, we are searching for a real solution to Laplace 's equation, and so our complex D'Alembert-type formula (3.55) is not entirely satisfactory. As we know, a complex number $z = x + \mathrm{i}y$ is real if and only if it equals its own conjugate, $z = \overline{z}$. Thus, the solution (3.55) will be real if and only if

$$f(x + \mathrm{i}x) + g(x - \mathrm{i}y) = \overline{f(x + \mathrm{i}y)} + \overline{g(x - \mathrm{i}y)}\,.$$

To equate the two sides of this equation, we should require

$$g(x - \mathrm{i}y) = \overline{f(x + \mathrm{i}x)}$$

and so

$$u(x,y) = f(x + \mathrm{i}y) + \overline{f(x + \mathrm{i}y)} = 2\mathrm{Re}f(x + \mathrm{i}y).$$

Our main objectives goals of study are complex-valued functions $f(z)$ depending on a single complex variable $z = x + \mathrm{i}y$. In general, the

function $\mathrm{f}:\Omega\to\mathbb{C}$ will be defined on a sub-domain $\mathrm{z}\subset\Omega\subset\mathbb{C}$ of the complex plane. Any complex function can be written as

$$f(z)=f(x+\mathrm{i}y)=u(x,y)+\mathrm{i}v(x,y)\ ,$$

where , $u(x,y)=\mathrm{Re}f(z)$ is the real part of f , and $v(x,y)=\mathrm{Im}f(z)$ is the imaginary part of f .

Theorem 3.9.1

A function $f(z)=u(x,y)+\mathrm{i}v(x,y)$, where $z=x+\mathrm{i}y$, is analytic at z if and only if its real and imaginary parts are continuously differentiable and satisfy the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\ ,\qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\ , \tag{3.57}$$

In this case, the complex derivative of $f(z)$ is equal to any of the following expressions:

$$f'(z)=\frac{\partial f}{\partial x}=\frac{\partial u}{\partial x}+\mathrm{i}\frac{\partial v}{\partial x}=-\mathrm{i}\frac{\partial f}{\partial y}=\frac{\partial v}{\partial y}-\mathrm{i}\frac{\partial u}{\partial y}\ . \tag{3.58}$$

3.9.1 Harmonic Functions.

We began by motivating the analysis of complex functions through applications to the solution of the two-dimensional Laplace equation. Let us now formalize the precise relationship between the two subjects.

Theorem 3.9.2

The real and imaginary parts of an analytic function are harmonic. They are harmonic conjugate to each other, (i.e. $u(x,y)$,$v(x,y)$, are both harmonic functions).

Proof: Differentiating the Cauchy–Riemann equations (3.57), and invoking the equality of mixed partial derivatives, we find that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2} \ .$$

Therefore, u is a solution to the Laplace equation $u_{xx} + u_{xx} = 0$. The proof for v is similar.

3.9.2 Analytical Mapping

The intimate connections between complex analysis and solutions to the Laplace equation inspires us to look at changes of variables defined by complex functions. To this end, we will re-interpret a complex analytic function

$$\zeta = g(z) \quad \text{or} \quad \xi + i\eta = p(x,y) + \mathrm{i}\, q(x,y). \tag{3.58}$$

as a mapping that takes a point $z = x + \mathrm{i}y$ belonging to a prescribed domain $\Omega \subset \mathbb{C}$ to a point $\zeta = \xi + \mathrm{i}\,\eta$ belonging to the image domain $D = g(\Omega) \subset \mathbb{C}$. In many cases [18], the image domain D is the unit disk. In order to unambigouously relate functions on Ω to functions on D , we require that the analytic mapping (3.10.5.58) be one-to-one so that each

point $\zeta \in \mathrm{D}$ comes from a unique point $z \in \Omega$. As a result, the inverse function $z = g^{-1}(\zeta)$ is a well-defined map from D back to Ω, which we assume is also analytic on all of D . The calculus formula for the derivative of the inverse function

$$\frac{d}{d\zeta} g^{-1}(\zeta) = \frac{1}{g'(z)} \qquad \text{at } \zeta = g(z),$$

which remains valid for complex functions, implies that the derivative of $g'(z)$ must be nonzero everywhere in order that $g^{-1}(\zeta)$ be differentiable. This condition,

$$g'(z) \neq 0 \qquad \text{at every point} \qquad z \in \Omega$$

Finally, in order to match the boundary conditions, we will assume that the mapping extends continuously to the boundary $\partial\Omega$ and maps it, one-to-one, to the boundary $\partial\mathrm{D}$ of the image domain.

Proposition 3.9.1

If $w = f(z)$ is an analytic function of the complex variable $z = x + \mathrm{i}y$, and $\zeta = g(w)$ is an analytic function of the complex variable $w = u + \mathrm{i}v$, then the composition $\zeta = h(z) \equiv g \circ f(z) = g(f(z))$ is an analytic function of z .

3.9.3 Applications of Conformal Mapping to Harmonic Functions and Laplace's Equation:

We are interested in solving a boundary value problem for the Laplace equation on a domain $\Omega \subset \mathrm{R}^2$. Our strategy is to map it to a corresponding boundary value problem on the unit disk D [18] . To this end, suppose we know a conformal map $\zeta = \mathrm{g}(\mathrm{z})$ that takes $z \in \Omega$ to $\zeta \in \mathrm{D}$. As we know, the real and imaginary parts of an analytic function $f(\zeta)$ defined on D are harmonic. Moreover, according to Proposition (3.10.3), the composition $f(\mathrm{z}) = F(g(\mathrm{z}))$ defines an analytic function whose real and imaginary parts are harmonic functions on Ω. Thus, the conformal mapping can be regarded as a change of variables between their harmonic real and imaginary parts.

Proposition 3.9.2

If $U(\xi,\eta)$ is a harmonic function of ξ, η and

$$\zeta = \xi + \mathrm{i}\,\eta = \xi(x, y) + \mathrm{i}\,\eta(x, y) = g(z)$$

is any analytic function, then the composition

$$u(x, y) = U\big(\xi(x, y),\, \eta(x, y)\big)$$

is a harmonic function of x , y .

We conclude that whenever $U(\xi, \eta)$ is any harmonic function, and so solves the Laplace equation (in the ξ, η variables), then $u(x, y)$ is a

solution to the Laplace equation $\Delta u = 0$ in the x , y variables, and is thus also harmonic.

Example (3.5): Using the Argument Function

Consider the Dirichlet problem $\nabla^2 u = 0$ in the half plane $y > 0$ [18] , given the boundary values

$$u(x,0) = \begin{cases} 100 & \text{if } x > 0, \\ 50 & \text{if } x < 0. \end{cases}$$

Solution:

since the boundary condition is constant on the rays $x \geq 0$ and $x \leq 0$, it is reasonable to expect that the solution be constant on rays in the upper half plane. We try for a solution the function

$$u(x,y) = a \operatorname{Arg} z + b,$$

where a and b are real numbers and $z = x + iy$. The function is harmonic in the upper half-plane , its values on the boundary are $u(x, 0) = b$ if $x > 0$

and $u(x,0) = a\pi + b$ if $x < 0$. Thus, to satisfy the boundary conditions, take

$$b = 100 \text{ and } a\pi + 100 = 50, \text{ so } a = -\frac{50}{\pi} .$$

Hence

$$u(x,y) = -\frac{50}{\pi}\operatorname{Arg} z + 100$$

In terms of x and y, we can use $\operatorname{Arg}(z) = \cot^{-1}(\frac{x}{y})$ and get

$$u(x,y) = -\frac{50}{\pi}\cot^{-1}(\frac{x}{y}) + 100.$$

As $y \to 0^{+}, \cot^{-1}(\frac{x}{y})$ tends to 0 if $x > 0$ and π if $x < 0$, which shows that u satisfies the boundary condition.

Example (3.6): Dirichlet Problem in The First Quadrant

Consider the Dirichlet problem in the first quadrant Ω as shown in figure (3.4)

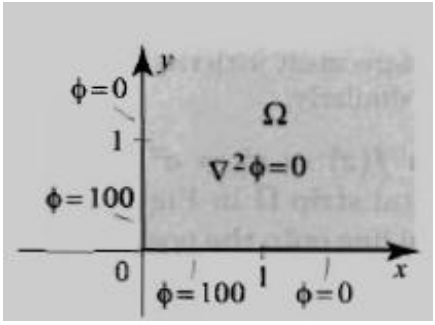


Figure (3.4): Solving the Dirichlet problem in the first quadrant

Solution:

We use the method of conformal mapping to transform the given problem into a problem on the upper half plane [4] . We choose $f(z) = z^2$ takes Ω in the z plane onto the upper half of the w plane figure (3.5). Moreover , the boundary of Ω is mapped onto the boundary of the upper half plane as

follows, the nonnegative real line $(x \geq 0)$ is mapped onto the nonnegative $(u \geq 0)$ real line, and the imaginary semi-axis iy with $(y \geq 0)$ is mapped to the non-positive $(u \leq 0)$. Now we will describe the boundary function in the Dirichlet problem in the w – plane . The boundary function in w – plane is $b \circ f^{-1}(w)$, where $b(z)$ is the boundary function in the z – plane [3]. With the help of figure(3.5) we see that $b \circ f^{-1}((u,0)) = 0$ if $|u| > 1$and $b \circ f^{-1}((u,0)) = 100$ if $|u| < 1$. The transformed Dirichlet problem in the upper half plane is described by figure (3.5) and given by:

$\nabla^2 U = 0$, w in the upper half plane.

$$U(u,0) = 0, \ |u| > 1, \quad U(u,0) = 100, \ |u| < 1.$$

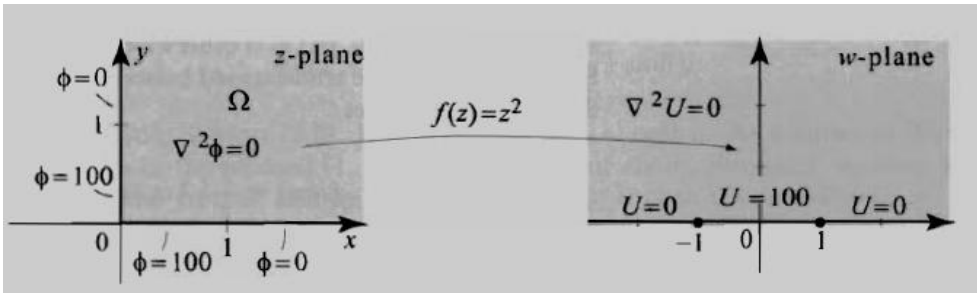


Figure (3.5): Transforming a Dirichlet problem from the first quadrant onto the upper half-plane. Notice the boundary correspondence

To solve the boundary value problem in the w – plane , we obtain

$$U(w) = \frac{100}{\pi}(\mathrm{Arg}(w-1) - \mathrm{Arg}(w+1)).$$

The solution to the original Dirichlet problem in the z -plane is

$$\phi(z) = U(f(z)) = \frac{100}{\pi}\left[\mathrm{Arg}(z^2-1) - \mathrm{Arg}(z^2+1)\right].$$

In terms of x and y , we have

$$z^2-1=x^2-y^2-1+2\mathrm{i}xy,$$

and $$z^2+1=x^2-y^2+1+2\mathrm{i}xy \quad .$$

Since the imaginary parts of z^2-1 and z^2+1 are positive, we use the inverse cotangent, and get

$$\phi(x,y)=\frac{100}{\pi}\left[\mathrm{Arg}(x^2-y^2-1+2\mathrm{i}xy)-\mathrm{Arg}(x^2-y^2+1+2\mathrm{i}xy)\right]$$

$$=\frac{100}{\pi}\left[\cot^{-1}\left(\frac{x^2-y^2-1}{2xy}\right)-\cot^{-1}\left(\frac{x^2-y^2+1}{2xy}\right)\right].$$

We will quickly verify some of the boundary conditions. If

$0<x<1$ and $y\to 0^+$, then

$$\frac{x^2-y^2-1}{2xy}\to-\infty \quad \text{and} \quad \frac{x^2-y^2+1}{2xy}\to+\infty \ .$$

Hence

$$\lim_{y\to 0^+}\left[\cot^{-1}\left(\frac{x^2-y^2-1}{2xy}\right)-\cot^{-1}\left(\frac{x^2-y^2+1}{2xy}\right)\right]=\pi-0=\pi \ .$$

and so $\lim_{y\to 0^+}\phi(\mathrm{x},\mathrm{y})=100$ if $0<x<1$, which is in agreement with the boundary condition.

Chapter Four

Finite Difference Methods for Maxwell's Equations

4. Finite Difference Methods for Maxwell's Equations

In this chapter we will try to solve the Maxwell's equations by well-known advanced numerical methods. Namely the Finite Difference Method and the Finite Time Domain Method.

4.1 Introduction to Finite Difference Methods

The finite difference method (FDM) was first developed by A.Thom in the 1920_s under the title “the method of squares” to solve nonlinear hydrodynamic equations. Since then, the method has found applications in solving different field problems [26]. The finite difference techniques are based upon approximations which permit replacing differential equations by finite difference equations. These finite difference approximations are algebraic in form; they relate the value of the dependent variable at a point in the solution region to the values at some neighboring points. Thus a finite difference solution basically involves three steps [26]:

- dividing the solution region into a grid of nodes.
- approximating the given differential equation by finite difference equivalent that relates the dependent variable at a point in the solution region to its values at the neighboring points.
- solving the difference equations subject to the prescribed boundary conditions and/or initial conditions.

Maxwell's equations are usually formulated as differential equations. Therefore, it is quite natural to solve them by finite difference methods. This essentially involves estimating derivatives numerically, where the derivatives are approximated by differences between neighboring points on a grid.

4.2 Finite Difference Scheme for The Wave Equation

We will focus on the equation for $\mathbf{E}$; everything we do will obviously pertain to the $\mathbf{B}$ equation as well. Furthermore, we will simplify things initially by imagining that $\mathbf{E}$ only depends on x and t. The equation we derived in chapter two section (2.7) for $\mathbf{E}$ then reduces to

$$c^2 \frac{\partial^2 \mathbf{E}}{\partial x^2} = \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{4.1}$$

At this point, it is worth taking a brief detour to talk about equations of this form more generally. This equation for the electric field is a special case of the wave equation in general

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

where c is the speed of the wave [26]. An equivalent finite difference formula is

$$c^2 \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta t)^2}$$

where $x = i\Delta x,\ t = j\Delta t,\ i, j = 0,1,2,\ldots$

This equation can be written as

$$u_{i,j+1} = 2(1-r)\,u_{i,j} + r[u_{i+1,j} + u_{i-1,j}] - u_{i,j-1} \tag{4.2}$$

where $u_{i,j}$ is an approximation to $u(x,t)$ and r is the "aspect ratio" [26] given by

$$r = \left(\frac{c\Delta t}{\Delta x}\right)^2 \tag{4.3}$$

Equation (4.2) is an explicit formula for the wave equation. For the solution algorithm in Eq.(4.2) to be stable, the aspect ratio must be $r \le 1$, If we choose $r = 1$, Eq. (4.2) becomes

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1} \tag{4.4}$$

The two-step schemes of equations (4.2) and (4.4) require that the values of u at times j and $j-1$ be known to get u at time $j+1$. Thus, we must derive a separate algorithm to "start" the solution of Eq. (4.2) or Eq.(4.4); that is, we must compute $u(i,1)$ and $u(i,2)$. To do this, we utilize the prescribed initial condition. For example, suppose the initial condition on the PDE in Eq.(4.1) is

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0\,.$$

We use the centered-difference formula

$$\frac{\partial u(x,0)}{\partial t} \simeq \frac{u_{i,1} - u_{i,-1}}{2\Delta t} = 0$$

or

$$u_{i,1} = u_{i,-1} \tag{4.5}$$

Substituting Eq. (4.5) into Eq. (4.2) and taking $j = 0$ (i.e., at $t = 0$), we get

$$u_{i,1} = (1-r)\, u_{i,0} + \frac{r}{2}[u_{i-1,0} + u_{i+1,0}] \tag{4.6}$$

Using the starting formula in Eq. (4.6) together with the prescribed boundary and initial conditions, the value of $u(x,t)$ at any grid point $u_{i,j}$ can be obtained directly from Eq. (4.2).

An application on this formula is the following example on the one dimensional initial boundary value problem;

Example (4.1)

We consider the one-dimensional wave equation

$$u_{tt} = u_{xx} \quad , \quad 0 < x < 1, \quad t \geq 0$$

subject to the boundary conditions

$$u(0,t)=0=u(1,t),\quad t\geq 0$$

and the initial conditions

$$u(x,0)=\sin\pi x,\quad 0<x<1,$$

$$u_t(x,0)=0,\quad 0<x<1,$$

Solution:

This IBV problem has already been solved in chapter 3 section (3.5.1), and its exact solution is given as:

$$u(x,t)=\sin\pi x\,\cos\pi t \tag{4.7}$$

Using the explicit finite difference scheme of Eq. (4.2) with $r=1$, we obtain the finite difference equation

$$u_{i,j+1}=u_{i-1,j}+u_{i+1,j}-u_{i,j-1},\quad j\geq 1 \tag{4.8}$$

For $j=0$, substituting

$$u_t=\frac{u_{i,1}-u_{i,-1}}{2\Delta t}=0 \tag{4.9}$$

or $$u_{i,1}=u_{i,-1}$$

into Eq. (4.8) gives the starting formula

$$u_{i,1}=\frac{1}{2}[\,u_{i-1,0}+u_{i+1,0}]$$

Since $c = 1.0$, and $r = 1.0$, $\Delta t = \Delta x$. Also, since the problem is symmetric with respect to $x = 0.5$, we solve for $u_{i,j}$ using Eqs. (4.8) and (4.9) within $0 < x < 0.5,\ t \geq 0$. The result shown in Table 4.1 is obtained for $\Delta t = \Delta x = 0.1$. The finite difference solution agrees with the exact solution in Eq. (4.7) to six decimal places [26]. The accuracy of the Finite Difference solution can be increased by choosing a smaller spatial increment Δx and a smaller time increment Δt .

Table 4.1: FD Solution of the Wave Equation.

x	0	0.1	0.2	0.3	0.4	0.5	0.6	
t								…
0	0	0.309	0.5879	0.889	0.9511	1	0.9511	…
0.1	0	0.2939	0.559	0.7694	0.9045	0.9511	0.9045	…
0.2	0	0.25	0.4755	0.6545	0.7694	0.809	0.7694	…
0.3	0	0.1816	0.3455	0.4755	0.559	0.5878	0.559	…
0.4	0	0.0955	0.1816	0.25	0.2939	0.309	0.2939	…
0.5	0	0	0	0	0	0	0	0
0.6	0	-0.0955	-0.1816	-0.25	-0.2939	-0.309	-0.2939	…
0.7	0	-0.1816	-0.3455	-0.4755	-0.559	-0.5878	-0.559	…

4.3 Finite Difference Solution of Laplace Equation

Here, we will apply the Finite Difference method for solving Laplace and Poisson equations, because various physical phenomenon are governed by these well-known equations in physical and engineering applications such as: steady heat condition, seepage through porous media [27], distributional potential etc. First we consider the Laplace equation in two dimensions

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{4.10}$$

We take a rectangular region R for which $u(x, y)$ is known at the boundary, (i.e. , Dirichlet boundary conditions) . We divide this region into a network of square mesh of side h as shown in figure (4.1).

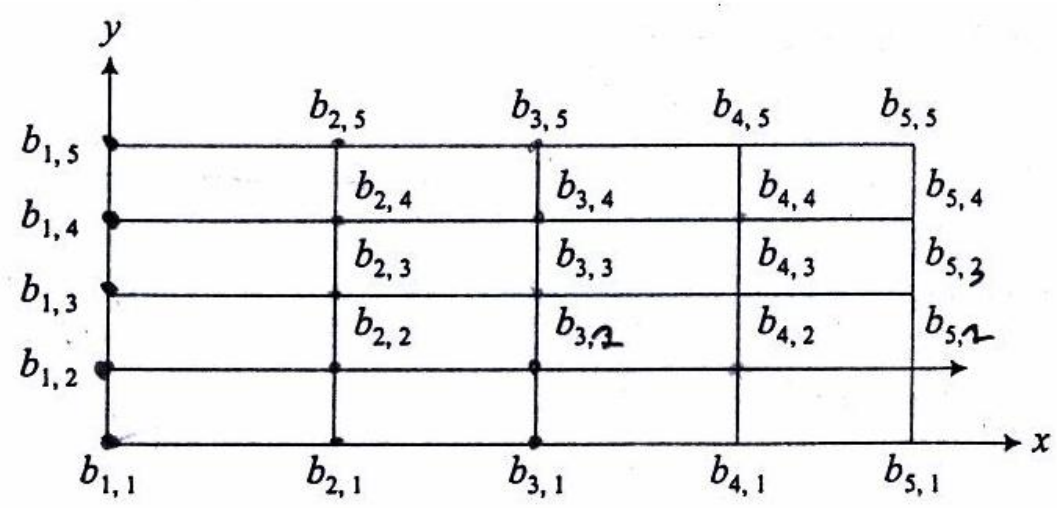


Figure (4.1): FD Solution to Laplace and Poisson equation with 5-points star

Replacing the derivatives in (4.10) by their finite difference approximation, we have

$$\frac{1}{h^2}[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}] + \frac{1}{h^2}[u_{i,j-1} - 2u_{i,j} + u_{i,j+1}] = 0$$

$$u_{i,j} = \frac{1}{4}[u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1}], \tag{4.11}$$

Equation (4.11) is known as *standard 5-point formula*. Sometimes we may use another formula

$$u_{i,j} = \frac{1}{4}[u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}], \tag{4.12}$$

This formula is called *diagonal 5-point formula*. Equation (4.12) serves as reasonably good approximation for obtaining starting values at the mesh points. We use the diagonal five-point formula in (4.12) to find the initial values of u at the interior mesh points and compute $u_{3,3}$, $u_{2,4}$, $u_{4,4}$, $u_{4,2}$ and $u_{2,2}$ in this order. We have

$$u_{3,3} = \frac{1}{4}(b_{1,5} + b_{5,1} + b_{5,5} + b_{1,1}),$$

$$u_{2,4} = \frac{1}{4}(b_{1,5} + b_{3,3} + b_{3,5} + b_{1,3}),$$

$$u_{4,4} = \frac{1}{4}(b_{3,5} + b_{5,5} + b_{5,3} + b_{3,3}),$$

$$u_{2,2} = \frac{1}{4}(b_{1,3} + b_{3,1} + b_{3,3} + b_{1,1}).$$

$$u_{2,4} = \frac{1}{4}(b_{1,5} + b_{3,3} + b_{1,3} + b_{3,5})$$

The values at the remaining interior points i.e. $u_{2,3}, u_{3,4}, u_{4,3}$ and $u_{3,2}$ are computed by the standard 5-point formula. Thus, we obtain

$$u_{2,3} = \frac{1}{4}(b_{1,3} + b_{3,3} + b_{2,4} + b_{2,2})$$

$$u_{3,4} = \frac{1}{4}(b_{2,4} + b_{4,4} + b_{3,5} + b_{3,3})$$

$$u_{4,3} = \frac{1}{4}(b_{3,3} + b_{5,3} + b_{4,4} + b_{4,2})$$

$$u_{3,2} = \frac{1}{4}(b_{2,2} + b_{4,2} + b_{3,3} + b_{3,1}).$$

After determining $u_{i,j}$ once, their accuracy is improved either by using Jacobi's iterative method or using Gauss-Seidel iterative method. But, it can be shown that the Gauss-Seidel scheme converges [27] approximately twice as fast as Jacobi's method. The process is repeated till the error ≤ a given tolerance. Here we have Jacobi's Iterative Method Formula is

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n)} + u_{i+1,j}^{(n)} + u_{i,j+1}^{(n)} + u_{i,j-1}^{(n)}\right]$$

and Gauss-Seidel Iterative Method Formula is

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j+1}^{(n+1)} + u_{i,j-1}^{(n)}\right]$$

4.4 Finite Difference Solution of Poisson Equation

Now, we consider the Poisson equation in two dimensions

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \tag{4.13}$$

The method of solving equation (4.13) is similar to that of Laplace equation (4.10). Here the standard 5-point formula for (4.13) taken the form

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f_{i,j} \tag{4.14}$$

Using (4.14) at each interior mesh point, we arrive at a system of linear equations in the nodal values $u_{i,j}$, which can be solved by Gauss-Seidel method.

Example (4.2) Application of Finite Difference Method for an Electrostatic Potential Problem (Laplace Equation)

Given the values of electrostatic potential $u(x,y)$ on the boundary of the square as shown in figure (4.2), we can evaluate the function $u(x,y)$ satisfying Laplace's equation $\nabla^2 u = 0$ at the mesh points of figure (4.2) by Jacobi's method [27].

Solution:

For initial values of u_1, u_2, u_3, u_4, we first assume that $u_4 = 0$ then we obtain

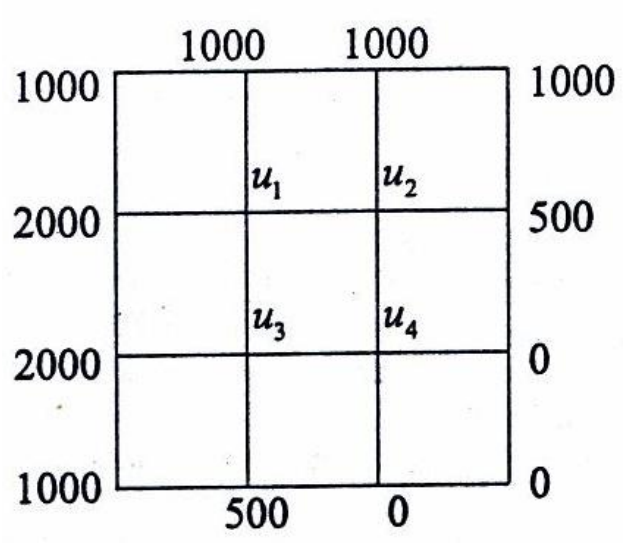


Figure (4.2): Solution to an electrostatic potential (Laplace Equation)

$$u_1 = \frac{1}{4}(1000+0+1000+2000) = 1000 \quad \text{(by diagonal formula)}$$

$$u_2 = \frac{1}{4}(1000+500+1000+0) = 625 \quad \text{(by standard formula)}$$

$$u_3 = \frac{1}{4}(2000+0+1000+500) = 875 \quad \text{(by standard formula)}$$

$$u_4 = \frac{1}{4}(875+0+625+0) = 375 \qquad \text{(by standard formula)}$$

Upon using the Jacobi's method

$$u_1^{(n+1)} = \frac{1}{4}(2000+u_2^{(n)}+1000+u_3^{(n)}),$$

$$u_2^{(n+1)} = \frac{1}{4}(u_1^{(n)}+500+1000+u_4^{(n)}),$$

$$u_3^{(n+1)} = \frac{1}{4}(2000+u_4^{(n)}+u_1^{(n)}+500),$$

$$u_4^{(n+1)} = \frac{1}{4}(u_3^{(n)}+0+u_2^{(n)}+0),$$

We carry out successive iteration using the above formula as given in the table below.

Table 4.2: FD Solution of Laplace Equation.

Iteration (n)	u_1	u_2	u_3	u_4
1	1125	719	969	375
2	1172	750	1000	422
3	1188	774	1024	438
4	1200	782	1032	450
5	1204	788	1038	454
6	1206.5	790	1040	456.5
7	1208	791	1041	458
8	1208	791.5	1041.5	458

Since there is no significant difference between the seventh and eight iteration values, therefore, we obtain:

$$u_1 = 1208, \quad u_2 = 792, \quad u_3 = 1042, \quad u_4 = 458,$$

This is the required solution.

Example (4.3) Application of Finite Difference to an Electrostatic Potential Problem (Poisson Equation)

Given the values of electrostatic potential on the boundary of the square in as shown in figure (4.3), we want to evaluate the function $u(x,y)$ satisfying Poisson equation [27]

$$\nabla^2 u = -10(x^2 + y^2 + 10)$$

over the square with the boundary conditions sides $(0 \le x \le 3)$ and $(0 \le y \le 3)$ on the boundary and mesh size $= 1.0$ by the Gauss-Seidel method.

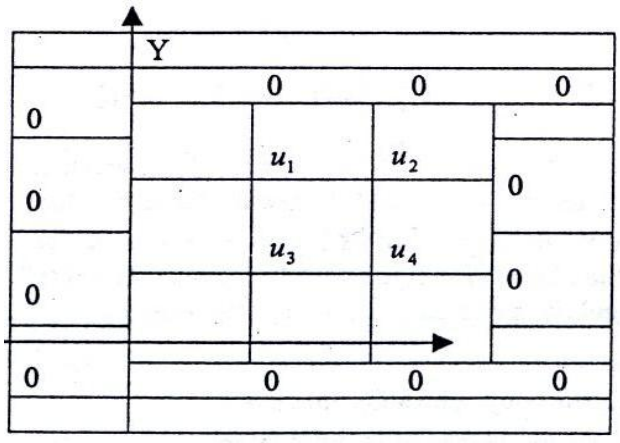


Figure (4.3): Solution to an electrostatic potential (Poisson Equation)

Solution:

here $h = 1$, therefore the standard 5-point formula for the given equation is given by

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = -10(i^2 - j^2 + 10) \qquad (4.15)$$

For $u_1(i=1, j=2)$ from (4.15), we get

$$0+u_2+u_3-4u_1=-10 \quad (1+4=10)$$

$$\Rightarrow \qquad u_1=\frac{1}{4}(u_2+u_3+150)$$

For $u_2(i=2, j=2)$ formula (4.15) gives us

$$u_2=\frac{1}{4}(u_1+u_4+180)$$

For $u_3(i=1, j=1)$, we get

$$u_3=\frac{1}{4}(u_1+u_4+120)$$

For $u_4(i=2, j=1)$, we obtain

$$u_4=\frac{1}{4}(u_2+u_3+150)=u_1$$

Thus, the above equations reduce to

$$u_1=\frac{1}{4}(u_2+u_3+150),\ u_2=\frac{1}{2}(u_1+90),\ u_3=\frac{1}{2}(u_1+60)$$

Now we solve these equations by the Gauss-Seidel iteration method and the solution is given in table below.

Table 4.3: FD Solution of Poisson Equation.

Iteration (n)	u_1	u_2	u_3
1	37.5	64	49
2	66	78	63
3	73	82	67
4	75	82.5	67.5
5	75	82.5	67.5

Because there is no difference in the values of 4^{th} and 5^{th} iterations, therefore we have.

$$u_1 = 75, u_2 = 82.5, u_3 = 67.5, u_4 = 75$$

This is the required solution.

4.5 Finite Difference Time Domain (FDTD) Method and The Yee Algorithm

The Finite-Difference Time-Domain (FDTD) method was originally proposed by Kane S. Yee in the seminar paper published in 1966 [13,28]. Yee proposed a discrete solution to Maxwell's equations based on central difference approximations of the spatial and temporal derivatives of the curl-equations. The novelty of Yee's approach was the staggering of the electric and magnetic fields in both space and time in order to obtain second-order accuracy. Yee derived a full three-dimensional formulation, and he validated the method with two-dimensional problems. The basic Yee-algorithm is restricted to a regularly-spaced orthogonal grid. This is not amenable to high-fidelity modeling of very complex geometries.

4.5.1 The Yee Algorithm for Maxwell's Equations

We recall the Maxwell's equations stated in chapter 2, these equations are listed below [13];

Faraday's Law: $$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} - \mathbf{M} \tag{4.16}$$

Ampere's Law: $$\frac{\partial \mathbf{D}}{\partial t} = -\nabla \times \mathbf{H} - \mathbf{J} \tag{4.17}$$

Gauss's Laws: $$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = \rho^{*} \tag{4.18}$$

Continuity Equations: $$\nabla \cdot \mathbf{J} = -\frac{\partial}{\partial t}\rho$$

$$\nabla \cdot \mathbf{M} = -\frac{\partial}{\partial t}\rho^{*} \tag{4.19}$$

Where, using MKS units [28], $\mathbf{B}$ is the magnetic flux density (Wb/m^2), $\mathbf{D}$ is the electric flux density (C/m^2), $\mathbf{E}$ is the electric field intensity (V/m), $\mathbf{H}$ is the magnetic field intensity (A/m), $\mathbf{J}$ is the electric current density (A/m^2), $\mathbf{M}$ is the magnetic current density (V/m^2), ρ is the electric charge density (C/m^3), and ρ^{*} is the magnetic charge density (Wb/m^3). The flux densities and the field intensities are related through the constitutive relations. For linear, isotropic media, these are:

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 \varepsilon_r \mathbf{E} \tag{4.20}$$

$$\mathbf{B} = \mu \mathbf{H} = \mu_0 \mu_r \mathbf{H} \tag{4.21}$$

where ε_0 is the free-space permittivity (8.854 × 10−12 F/m), ε_r is the relative permittivity, and ε is the permittivity (F / m) of the media. Similarly, μ_0 is the free-space permeability (4π × 10−7 H/m), μ_r is the relative permeability, and μ is the permeability (H / m) of the media.

4.5.2 The Yee-Algorithm

In 1966, Kane S. Yee derived an elegant, yet simple, time-dependent solution of Maxwell's equations based on their differential form using central difference approximations of both the space and the time-derivatives [13,28]. The formulation is based on discretizing the volume domain with a regular, structured, staggered, rectangular grid. Yee discovered that in order to maintain second-order accuracy of the central difference operators, the electric and magnetic fields must be staggered in both space and time. The novel scheme he derived to achieve this, now referred to as the Yee-algorithm, is detailed in this section. Consider a uniformly spaced rectangular grid in three-dimensions. Each grid cell has dimensions Δx, Δy, and Δz along each Cartesian axis. The coordinate of a node of the grid can be expressed in discrete form as: $(x, y, z)_{i,j,k} = (i\Delta x,\ j\Delta y,\ k\Delta z)$, where i, j, and k are integers. Similarly, the time is uniformly discretized as $t = n\Delta t$ .An arbitrary function $f(x, y, z, t)$ can be expressed at any node within the discrete space using the notation:

$$f(x,y,z,t) = f(i\Delta x, j\Delta y, k\Delta z, n\Delta t) = f^{n}_{i,j,k}. \qquad (4.22)$$

Within this uniform grid, the projections of the vector electric field parallel to a grid edges are sampled at edge grid edge center. Dual to this, the projection of the magnetic field normal to each grid cell face is sampled at the center of a grid face. This is illustrated in figure (4.4). Observing figure (4.4), it is apparent that the tangential electric field projected on the edges bounding a cell face circulate about the normal magnetic field vectors.

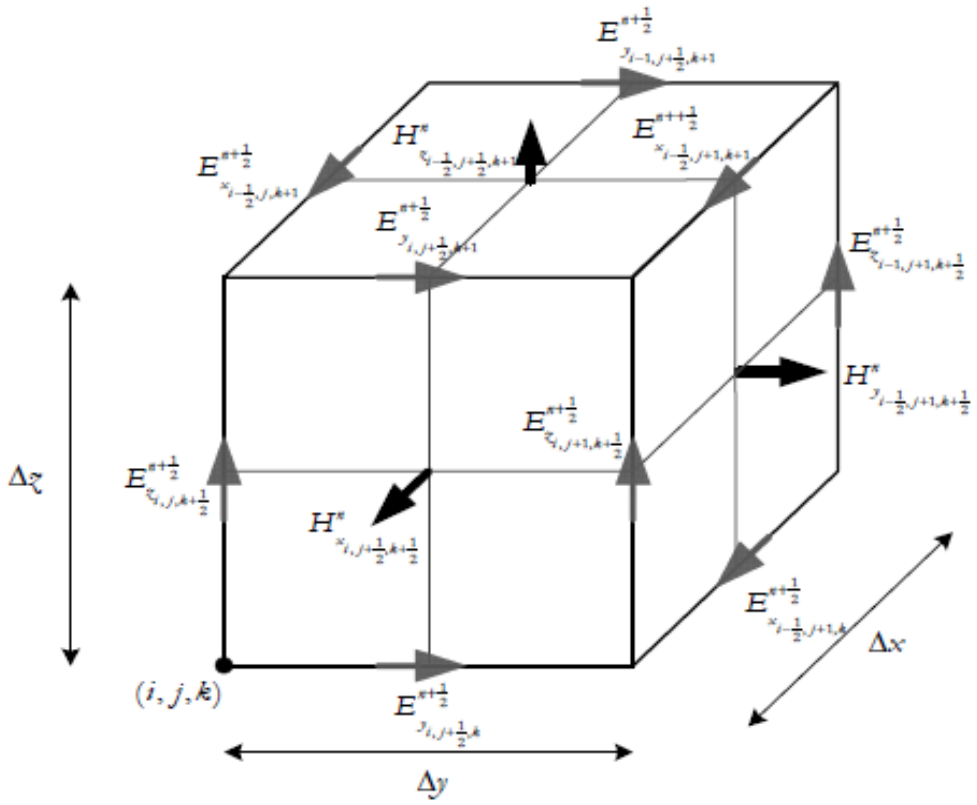


Figure (4.4): Primary grid cell of the regular, structured, rectangular, staggered grip.

This provides the essential pieces to formulate a curl operation. To illustrate this, consider the *x*-projection of Faraday's law in (4.16):

$$\mu \frac{\partial H_x}{\partial t} = \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} - M_x$$

where a linear isotropic material is assumed. Using the discretization of fig. (4.4), the time derivative and the spatial derivatives from the curl operator are approximated via central differences, leading to a discrete form of Faraday's law:

$$\mu\left(\frac{H_{x_{i,j+\frac{1}{2},k+\frac{1}{2}}}^{n+1} - H_{x_{i,j+\frac{1}{2},k+\frac{1}{2}}}^{n}}{\Delta t}\right) = \left(\frac{E_{y_{i,j+\frac{1}{2},k+1}}^{n+\frac{1}{2}} - E_{y_{i,j+\frac{1}{2},k}}^{n+\frac{1}{2}}}{\Delta z}\right)$$

$$-\left(\frac{E_{z_{i,j+1,k+\frac{1}{2}}}^{n+\frac{1}{2}} - E_{z_{i,j,k+\frac{1}{2}}}^{n+\frac{1}{2}}}{\Delta y}\right) - M_{x_{i,j+\frac{1}{2},k+\frac{1}{2}}}^{n+\frac{1}{2}}. \qquad (4.23)$$

It is observed from (4.23) that the central difference approximations of both spatial derivatives of the electric field projections are second-order accurate at the face center, which is the sample location of the normal magnetic field. By staggering the magnetic field and the electric field in time, the time-derivative is also second-order accurate. Consequently, the difference operator in (4.23) is second-order accurate in both space and time.

Following the same procedure, one can derive similar expressions for the *y* and *z*-projections of Faraday's law as well:

$$\mu\left(\frac{H_{y_{i+\frac{1}{2},j,k+\frac{1}{2}}}^{n+1}-H_{y_{i+\frac{1}{2},j,k+\frac{1}{2}}}^{n}}{\Delta t}\right)=\left(\frac{E_{z_{i+1,j,k+\frac{1}{2}}}^{n+\frac{1}{2}}-E_{z_{i,j,k+\frac{1}{2}}}^{n+\frac{1}{2}}}{\Delta x}\right)$$

$$-\left(\frac{E_{x_{i+\frac{1}{2},j,k+1}}^{n+\frac{1}{2}}-E_{x_{i+\frac{1}{2},j,k}}^{n+\frac{1}{2}}}{\Delta z}\right)-M_{y_{i+\frac{1}{2},j,k+\frac{1}{2}}}^{n+\frac{1}{2}}, \qquad (4.24)$$

$$\mu\left(\frac{H_{z_{i+\frac{1}{2},j+\frac{1}{2},k}}^{n+1}-H_{z_{i+\frac{1}{2},j+\frac{1}{2},k}}^{n}}{\Delta t}\right)=\left(\frac{E_{x_{i+\frac{1}{2},j+1,k}}^{n+\frac{1}{2}}-E_{z_{i+\frac{1}{2},j,k}}^{n+\frac{1}{2}}}{\Delta y}\right)$$

$$-\left(\frac{E_{y_{i+1,j+\frac{1}{2},k}}^{n+\frac{1}{2}}-E_{y_{i,j+\frac{1}{2},k}}^{n+\frac{1}{2}}}{\Delta x}\right)-M_{z_{i+\frac{1}{2},j+\frac{1}{2},k}}^{n+\frac{1}{2}}. \qquad (4.25)$$

The discrete form of Ampère's law is derived via a secondary grid cell, as illustrated in figure (4.5). The secondary grid cell edges connect the cell centers of the primary grid cells illustrated in Fig. (4.4). The secondary grid cell also has dimensions Δx, Δy, and Δz. Thus, the edges of the secondary grid pass through the centers of the faces of the secondary grid cells. Dually, the edges of the primary grid

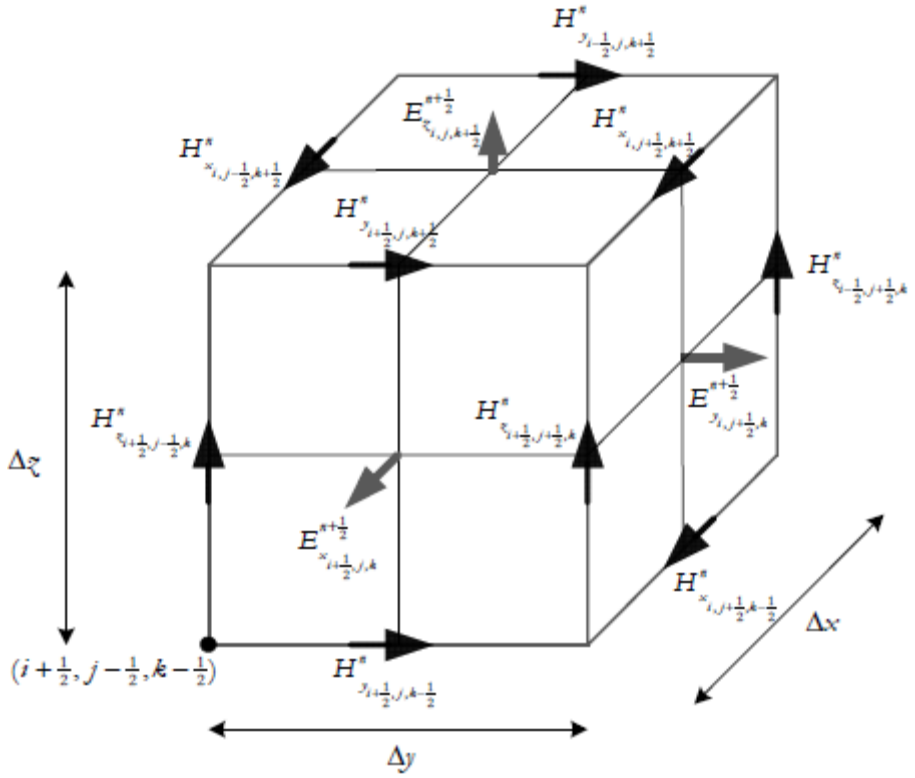


Figure (4.5): Secondary grid cell of the regular, structured, rectangular, staggered grip.

pass through the face centers of the secondary grid cells. Consequently, the electric and magnetic field vectors have dual roles in the primary and secondary grids. Observing a secondary grid face in Fig. (4.5), it is apparent that the magnetic field lines on a cell face circulates about the normal electric field line. Again, this provides the essential components of a curl operation. The *x*-projection of Ampère's law is expressed in a discrete form as:

$$\varepsilon\left(\frac{E_{x_{i+\frac{1}{2},j,k}}^{n+\frac{1}{2}}-E_{x_{i+\frac{1}{2},j,k}}^{n-\frac{1}{2}}}{\Delta t}\right)=\left(\frac{H_{z_{i+\frac{1}{2},j+\frac{1}{2},k}}^{n}-H_{z_{i+\frac{1}{2},j-\frac{1}{2},k}}^{n}}{\Delta y}\right)$$

$$-\left(\frac{H_{y_{i+\frac{1}{2},j,k+\frac{1}{2}}}^{n}-H_{y_{i+\frac{1}{2},j,k-\frac{1}{2}}}^{n}}{\Delta z}\right)-J_{x_{i+\frac{1}{2},j,k}}^{n}, \qquad (4.26)$$

where an isotropic, linear, lossless media has been assumed. Similar expressions can be derived for the *y* and *z*-projections of Ampère's law. The discrete form of Faraday's and Ampère's laws lead to a total of six equations, which can then be used to solve for the time-dependent vector field intensities. Yee proposed to do this with an explicit time-marching scheme. To this end, it is assumed that the initial values of the discrete fields are known over all space. Subsequently, a recursive solution scheme can be used to advance the fields through time. For example, from (4.23), assuming that E_y^{n+1},E_z^{n+1} and H_x^n are known at all spatial samples, an explicit update operator used to solve for H_x^{n+1} is expressed as:

$$H_{x_{i,j+\frac{1}{2},k+\frac{1}{2}}}^{n+1}=H_{x_{i,j+\frac{1}{2},k+\frac{1}{2}}}^{n}+\frac{\Delta t}{\mu}\left[\left(\frac{E_{y_{i,j+\frac{1}{2},k+1}}^{n+\frac{1}{2}}-E_{y_{i,j+\frac{1}{2},k}}^{n+\frac{1}{2}}}{\Delta z}\right)\right.$$

$$\left.-\left(\frac{E_{z_{i,j+1,k+\frac{1}{2}}}^{n+\frac{1}{2}}-E_{z_{i,j,k+\frac{1}{2}}}^{n+\frac{1}{2}}}{\Delta y}\right)-M_{x_{i,j+\frac{1}{2},k+\frac{1}{2}}}^{n+\frac{1}{2}}\right]. \qquad (4.27)$$

Similarly, we obtain

$$H_{y_{i+\frac{1}{2},j,k+\frac{1}{2}}}^{n+1} = H_{y_{i+\frac{1}{2},j,k+\frac{1}{2}}}^{n} + \frac{\Delta t}{\mu}\left[\left(\frac{E_{z_{i+1,j,k+\frac{1}{2}}}^{n+\frac{1}{2}} - E_{z_{i,j,k+\frac{1}{2}}}^{n+\frac{1}{2}}}{\Delta x}\right)\right.$$

$$\left.-\left(\frac{E_{x_{i+\frac{1}{2},j,k+1}}^{n+\frac{1}{2}} - E_{x_{i+\frac{1}{2},j,k}}^{n+\frac{1}{2}}}{\Delta z}\right) - M_{y_{i+\frac{1}{2},j,k+\frac{1}{2}}}^{n+\frac{1}{2}}\right]. \qquad (4.28)$$

$$H_{z_{i+\frac{1}{2},j+\frac{1}{2},k}}^{n+1} = H_{z_{i+\frac{1}{2},j+\frac{1}{2},k}}^{n} + \frac{\Delta t}{\mu}\left[\left(\frac{E_{x_{i+\frac{1}{2},j+1,k}}^{n+\frac{1}{2}} - E_{x_{i+\frac{1}{2},j,k}}^{n+\frac{1}{2}}}{\Delta y}\right)\right.$$

$$\left.-\left(\frac{E_{y_{i+1,j+\frac{1}{2},k}}^{n+\frac{1}{2}} - E_{y_{i,j+\frac{1}{2},k}}^{n+\frac{1}{2}}}{\Delta x}\right) - M_{z_{i+\frac{1}{2},j+\frac{1}{2},k}}^{n+\frac{1}{2}}\right]. \qquad (4.29)$$

$$E_{x_{i+\frac{1}{2},j,k}}^{n+\frac{1}{2}} = E_{x_{i+\frac{1}{2},j,k}}^{n-\frac{1}{2}} + \frac{\Delta t}{\varepsilon}\left[\left(\frac{H_{z_{i+\frac{1}{2},j+\frac{1}{2},k}}^{n} - H_{z_{i+\frac{1}{2},j-\frac{1}{2},k}}^{n}}{\Delta y}\right)\right.$$

$$\left.-\left(\frac{H_{y_{i+\frac{1}{2},j,k+\frac{1}{2}}}^{n} - H_{y_{i+\frac{1}{2},j,k-\frac{1}{2}}}^{n}}{\Delta z}\right) - J_{x_{i+\frac{1}{2},j,k}}^{n}\right]. \qquad (4.30)$$

$$E_{y_{i,j+\frac{1}{2},k}}^{n+\frac{1}{2}} = E_{y_{i,j+\frac{1}{2},k}}^{n-\frac{1}{2}} + \frac{\Delta t}{\varepsilon}\left[\left(\frac{H_{x_{i,j+\frac{1}{2},k+\frac{1}{2}}}^{n} - H_{x_{i,j+\frac{1}{2},k-\frac{1}{2}}}^{n}}{\Delta z}\right)\right.$$

$$\left. -\left(\frac{H_{z_{i+\frac{1}{2},j+\frac{1}{2},k}}^{n} - H_{z_{i-\frac{1}{2},j+\frac{1}{2},k}}^{n}}{\Delta x}\right) - J_{y_{i,j+\frac{1}{2},k}}^{n}\right]. \qquad (4.31)$$

$$E_{z_{i,j,k+\frac{1}{2}}}^{n+\frac{1}{2}} = E_{z_{i,j,k+\frac{1}{2}}}^{n-\frac{1}{2}} + \frac{\Delta t}{\varepsilon}\left[\left(\frac{H_{y_{i+\frac{1}{2},j,k+\frac{1}{2}}}^{n} - H_{y_{i-\frac{1}{2},j,k+\frac{1}{2}}}^{n}}{\Delta x}\right)\right.$$

$$\left. -\left(\frac{H_{x_{i,j+\frac{1}{2},k+\frac{1}{2}}}^{n} - H_{x_{i,j-\frac{1}{2},k+\frac{1}{2}}}^{n}}{\Delta y}\right) - J_{z_{i,j,k+\frac{1}{2}}}^{n}\right]. \qquad (4.32)$$

Equations (4.27)–(4.32) are the first-order difference equations defining Yee's algorithm and are the foundation of the FDTD method [4]. These equations provide an explicit recursive update scheme of the electromagnetic fields in linear, isotropic, lossless media throughout the entire volume.

4.5.3 Gauss's Laws

The discrete approximations of Maxwell's curl equations must also satisfy Gauss's laws [28]. If they do not, then spurious charge can corrupt the numerical solution. First consider Gauss's law for the magnetic field as presented in (4.18). Assuming a charge-free region, then:

$$\nabla \cdot \mathbf{B} = 0 \tag{4.33}$$

Must hold true if this is differentiated with respect to time, then:

$$\frac{\partial}{\partial t} \nabla \cdot \mathbf{B} = 0 \tag{4.34}$$

must also be true. The spatial derivative is then approximated with a central difference approximation, leading to a discrete form of Gauss's law:

$$\frac{\partial}{\partial t}\left(\frac{B_{x_{i,j+\frac{1}{2},k+\frac{1}{2}}}^{n+1} - B_{x_{i-1,j+\frac{1}{2},k+\frac{1}{2}}}^{n+1}}{\Delta x} + \frac{B_{y_{i+\frac{1}{2},j,k+\frac{1}{2}}}^{n+1} - B_{y_{i+\frac{1}{2},j-1,k+\frac{1}{2}}}^{n+1}}{\Delta y} \right.$$

$$\left. \frac{B_{z_{i+\frac{1}{2},j+\frac{1}{2},k}}^{n+1} - B_{z_{i+\frac{1}{2},j+\frac{1}{2},k-1}}^{n+1}}{\Delta z} \right) = 0. \tag{4.35}$$

Substituting in the discrete expressions for the right-hand-sides of (4.23)–(4.25) (with the *M*'s = 0), then it is found that the discrete electric fields identically cancel, and (4.35) is exactly satisfied . A dual expression can be derived for Gauss's law for the electric field. Thus, it is concluded that the discrete representation satisfies Gauss's law to the extent that the total charge density in the discrete system is constant. Therefore, if the initial charge density is zero, then the total charge density in the discrete system will remain zero, and Gauss's law is strictly satisfied.

4.6 The Five Point Star

As we have mentioned before, The first step in applying FDM is to define a mesh, which is simply a uniform grid of spatial points at which the voltage function will be sampled [23]. Letting h be the mesh size as shown in figure (4.6), the mesh points may be defined by

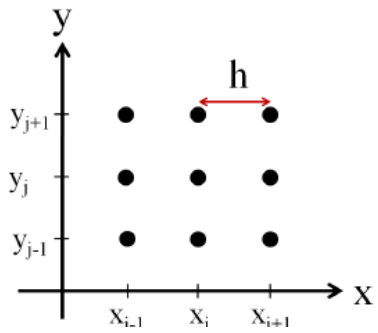


Figure (4.6): Mesh points for the FDM grid.

$$x_i = ih, \text{ and } y_j = jh$$

where i and j are integers. In practice, i and j will eventually be used as indices for a matrix of voltage samples, we shall therefore replace the spatial coordinates with simple indices by assuming the following convention:

$$V_{i,j} = V(x_i, y_j)$$

In a similar fashion, we may also define the charge density samples along the same mesh by using the $\rho(i,j)$ notation. The next step is to expand the Poisson equation by explicitly showing the partial derivatives in space:

$$\frac{\partial^2 V_{i,j}}{\partial x^2} + \frac{\partial^2 V_{i,j}}{\partial y^2} = -\frac{\rho_{i,j}}{\varepsilon_0} \ .$$

The reason for doing this is to approximate the derivative operators through the use of finite-differences. The easiest way to do this is through the three-point approximation for the second-derivative, which is given as

$$\frac{\partial^2}{\partial x^2} V_{i,j} \approx \frac{V_{i-1,j} - 2V_{i,j} + V_{i+1,j}}{h^2} \ ,$$

$$\frac{\partial^2}{\partial y^2} V_{i,j} \approx \frac{V_{i,j-1} - 2V_{i,j} + V_{i,j+1}}{h^2} \ ,$$

Finally, we solve for $V_{i,j}$ to find

$$V_{i,j} = \frac{1}{4}\left[V_{i-1,j} + V_{i+1,j} + V_{i,j-1} + V_{i,j+1} + \frac{\rho_{i,j} h^2}{\varepsilon_0} \right]$$

This expression tells us is that every voltage sample $V(i,j)$ is dependent only on $\rho(i,j)$ and the voltage at the four nearest neighbors. Because each voltage sample $V(i,j)$ is linearly dependent on its four nearest neighbors, the solution over all (i,j) may be represented as a simple matrix-vector equation. This is readily achieved by defining the vector $\mathbf{X}$ to contain all of the voltage samples within the domain. For example, one simple method might scan row-wise along the voltage samples according to the convention:

$$\mathbf{X} = [\ V(1,1)\ V(1,2)\ V(1,3) \ldots V(2,1) \ldots V(3,1) \ldots\ ]^T$$

The next step is to express the linear relationship between voltage samples into a matrix $\mathbf{A}$. This effectively converts the entire problem into a matrix-vector equation with the form

$$\mathbf{A}\mathbf{X} = \mathrm{b} \ ,$$

where b contains all the information about any charge densities and boundary conditions. The numerical solution to the system is finally found by simply inverting the matrix $\mathbf{A}$ to arrive at the solution

$$\mathbf{X} = \mathbf{A}^{-1}\mathrm{b} \ .$$

Example (4.4): A simple 4×4 grid

Consider the simple, 4×4 grid of voltage samples depicted in figure (4.7). The top boundary is a Dirichlet boundary fixed at 1.0 V with bottom boundary grounded at 0.0 V. The left and right boundaries are Neumann boundaries fixed to a derivative of 0.0 V/m with respect to the outward normal. Using FDM, it is our job to solve for the voltage potentials at all of the indicated points. The first step is to establish some sort of numbering convention so that the unknown vector $\mathbf{x}$ may be defined [23]. One straightforward way to do this is by scanning across the rows, as indicated by the numbering in figure (4.7), It is also worth emphasizing that the samples along the corners of the domain do not make any difference to the final solution of the problem with respect to the interior points. This is because neither the boundary conditions nor the five-point star will depend

on what values are placed within the corners. We will therefore neglect these points entirely from the solution set, though in practice it can often be easier to just assign convenient values to them. The vector of unknowns will therefore be written as

$$\mathbf{X} = [\ V_1\ V_2\ V_3\ \ldots\ V_{12}\ ]^T\ .$$

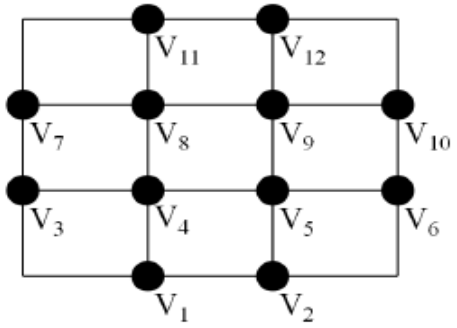


Figure (4.7): Sampled grid of voltages

The next step is to fill the system matrix $\mathbf{A}$. We begin by noting that V_1 and V_2 are both Dirichlet boundaries fixed at $0.0\,\mathrm{V}$. The first two rows in $\mathbf{A}$ are therefore nothing but zeros with a one placed at the diagonal element. The same is also true for V_{11} and V_{12} since these are likewise Dirichlet boundaries. The Neumann boundaries are filled in a similar manner, but with a -1 placed on the column corresponding to the interior point. For V_3 and V_7, this is the first element to the right of the diagonal. For V_6 and V_{10}, the -1 is placed at the first element to the left of the diagonal. For the remainder of the samples, the five-point star dictates a value of -4 to be placed at the diagonal, with four 1's placed at their corresponding columns that represent the neighboring points. This will include the two

columns immediately adjacent to the diagonal, plus two other 1's placed at the appropriate locations. The final step is to fill the forcing vector $\mathbf{b}$. Generally speaking, this will be a vector of all zeros except at the points where there is a nonzero boundary condition or a nonzero value for ρ. Thus, $\mathbf{b}$ has only two 1's placed in the last two rows, with zeros placed at all other elements. Writing out the full linear system $\mathbf{AX} = \mathrm{b}$, therefore leads to.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \\ V_9 \\ V_{10} \\ V_{11} \\ V_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Finally, we solve for $\mathbf{X}$ and get:

$$\mathbf{X} = [\ 0\ 0\ \tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{2}{3}\ \tfrac{2}{3}\ \tfrac{2}{3}\ \tfrac{2}{3}\ 1\ 1\ ]^T \text{ Volts}.$$

4.7 Accuracy, Convergence and Stability of FD Schemes

When using numerical tools, one must keep in mind that they never give the exact answer. The accuracy of the numerical result depends on the resolution. In general, one does not know the order of convergence of a

computational method for a given problem a priori [25]. Even though standard centered finite differences converge with an error of order h^2 (where h is the grid spacing or the cell size) for regular problems, Accuracy has to do with the closeness of the approximate solution to exact solutions (assuming they exist). Stability is the requirement that the scheme does not increase the magnitude of the solution with increase in time. There are three sources of errors that are nearly unavoidable in numerical solution of physical problems [25]:

- modeling errors,
- truncation (or discretization) errors, and
- round off errors.

Each of these error types will affect accuracy and therefore degrade the solution. The modeling errors are due to several assumptions made in arriving at the mathematical model. For example, a nonlinear system may be represented by a linear PDE. Truncation errors arise from the fact that in numerical analysis, we can deal only with a finite number of terms from processes which are usually described by infinite series. For example, in deriving finite difference schemes, some higher-order terms in the Taylor series expansion were neglected, thereby introducing truncation error. Round off errors reflect the fact that computations can be done only with a finite precision on a computer. This unavoidable source of errors is due to the limited size of registers in the arithmetic unit of the computer. Round

off errors can be minimized by the use of double-precision arithmetic. The only way to avoid round off errors completely is to code all operations using integer arithmetic. This is hardly possible in most practical situations. Although reducing the mesh size *h* will increase accuracy, but it is not possible to indefinitely reduce *h*. Decreasing the truncation error by using a finer mesh may result in increasing the round off error due to the increased number of arithmetic operations. A numerical algorithm is said to be stable if a small error at any stage produces a smaller cumulative error. It is unstable otherwise. To determine whether a finite difference scheme is stable, we define an error, ε^n, which occurs at time step n, assuming that there is one independent variable. We define the amplification of this error [25] at time step $n+1$ as

$$\varepsilon^{n+1} = g\,\varepsilon^n \tag{4.36}$$

Where g is known as the *amplification factor*. In more complex situations, we have two or more independent variables, and Eq. (4.36) becomes

$$[\varepsilon]^{n+1} = [G][\varepsilon]^n \tag{4.37}$$

where $[G]$ is the amplification matrix. For the stability of the difference scheme, it is required that Eq. (4.36) satisfy

$$\left|\varepsilon^{n+1}\right| \le \left|\varepsilon^{n}\right|$$

or

$$\left|g\right| \le 1 \tag{4.38a}$$

For the case in Eq. (4.37),

$$\left\|G\right\| \le 1 \tag{4.38b}$$

4.7.1 Convergence Analysis

The analytical solution $u(x_m, t_n)$ of differential equation, the difference solution u_m^n of the difference equation [27] and numerical solution $\bar{u}_m^n$ are related by the relation.

$$\left|u(x_m, t_n) - \bar{u}_m^n\right| \le \left|u(x_m, t_n) - u_m^n\right| + \left|u_m^n - \bar{u}_m^n\right| \tag{4.39}$$

The value $\left|u(x_m, t_n) - u_m^n\right|$ is called the local truncation error which arises by replacing the differential equation by the difference equation. The truncation error converges to zero as h and k both tends to zero for a convergent difference scheme. The difference $\left|u_m^n - \bar{u}_m^n\right|$ is known as numerical error which arises because in actual computations we can't solve the difference equation exactly due to round off error. If the error made at one stage of calculations don't cause increasingly with continued computations but will eventually damp out, then we say that solution is

stable. If the difference scheme is stable then second term in (4.39) practically equals zero. Thus the results of convergent and stable methods are very close to actual results.

4.7.2 Convergence, Consistency and Stability

Definition **(4.7.1)**

A one-step finite difference scheme approximating a partial differential equation is ***convergent*** scheme if for any solution to the partial differential equation, $u(x,t)$, and solutions to the finite difference scheme, v_i^n , such that v_i^0 converges to $u_0(x)$ as $i\Delta x$ converges to x , then v_i^n converges to $u(x,t)$ as $(i\Delta x, n\Delta t)$ converges to (x,t) as Δx , Δt converge to 0 .

Definition **(4.7.2)**

Given a partial differential equation $Pu = f$ and a finite difference scheme, $P_{\Delta x,\Delta t}u = f$, we say that the finite difference scheme is consistent with the partial differential equation if for any smooth function $\phi(x,t)$

$$P\phi - P_{\Delta x,\Delta t}\phi \to 0 \text{ as } \Delta x, \Delta t \to 0.$$

Example (4.5)

Consider the one-way wave equation given by the operator $P = \partial/\partial t + \alpha \partial/\partial x$

$$P\phi = \phi_t + \alpha \phi_x$$

with α greater than 0. We will evaluate the consistency of the forward-time forward-space scheme with difference operator $P_{\Delta x, \Delta t}$ given by

$$P_{\Delta x, \Delta t}\phi = \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} + \alpha \frac{\phi_{i+1}^n - \phi_i^n}{\Delta x}$$

We begin by taking the Taylor expansion of the function ϕ in t and x about (x_i, t_n). We have that

$$\phi_i^{n+1} = \phi_i^n + \Delta t \phi_t + \frac{1}{2}\Delta t^2 \phi_{tt} + O(\Delta t^3)$$

$$\phi_{i+1}^n = \phi_i^n + \Delta x \phi_x + \frac{1}{2}\Delta x^2 \phi_{xx} + O(\Delta x^3)$$

This gives us

$$P_{\Delta x, \Delta t}\phi = \phi_t + \alpha \phi_x + \frac{1}{2}\Delta t \phi_{tt} + \alpha \frac{1}{2}\Delta x \phi_{xx} + O(\Delta t^2) + O(\Delta x^2)$$

Thus

$$P\phi - P_{\Delta x, \Delta t}\phi = \frac{1}{2}\Delta t \phi_{tt} + \frac{1}{2}\Delta x \phi_{xx} + O(\Delta t^2) + O(\Delta x^2)$$

$$\to 0 \quad \text{as} \quad (\Delta x, \Delta t) \to 0.$$

Thus, this scheme is consistent.

4.7.3 Stability Analysis for 1D-wave Equation [26]

If we consider the following one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{4.40}$$

Its solution is the displacement function $u(x,t)$ defined for values of x from 0 to l and for t from 0 to ∞ , satisfying the initial and boundary conditions. Such equations arise from convective type of problems in vibrations, wave mechanics, gas dynamics, elasticity, electromagnetic and seismology.

We now consider the boundary value problem defined by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1 \tag{4.41}$$

Subject to the conditions

$$u(x,0) = f(x), \ \frac{\partial u}{\partial t}(x,0) = g(x), \ 0 \le x \le 1 \tag{4.42}$$

$$u(0,t) = \varphi(t), \ u(1,t) = \varphi(t), \ 0 \le t \le T \tag{4.43}$$

As previously, we use the difference approximations for the derivatives

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h^2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) + O(h^2) \tag{4.44}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{k^2}(u_{i,j-1} - 2u_{i,j} + u_{i,j+1}) + O(k^2) \tag{4.45}$$

Where $x = ih, \ \ i = 0,1,2,\ldots$ and $t = jk, \ \ j = 0,1,2,\ldots$

The derivative $\frac{\partial u}{\partial t}$ is approximated as

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j-1}}{2k} + O(k^2)$$

Using (4.42) and (4.43) in (4.41), we obtain

$$\frac{1}{k^2}(u_{i,j-1} - 2u_{i,j} + u_{i,j+1}) = \frac{c^2}{h^2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$$

$$\Rightarrow\ u_{i,j+1} = -u_{i,j-1} + \alpha^2(u_{i-1,j} + u_{i+1,j}) + 2(1-\alpha^2)\,u_{i,j} \qquad (4.46)$$

Where $\alpha = \frac{ck}{h}$. the formula (4.46) shows that the function values at j^{th} and $(j-1)^{th}$ levels are required in order to determine those at the $(j+1)^{th}$ level. Such difference schemes are called three level difference schemes. By expanding the terms in (4.46) as Taylor's series and simplifying. It can be shown that the truncation error is of order $O(k^2 + h^2)$ and the formula (4.46) holds good if $\alpha < 1$, which is the condition for stability. There also exist implicit finite difference schemes for the equation (4.40). two such schemes are

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}$$

$$= \frac{c^2}{2h^2}\left[u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}\right] + \left[u_{i+1,j-1} - 2u_{i,j-1} + u_{i-1,j-1}\right] \qquad (4.47)$$

and

$$\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{k^2}=\frac{1}{4h^2}\left[(u_{i+1,j+1}-2u_{i,j+1}+u_{i-1,j+1})\right.$$

$$\left.+2(u_{i+1,j}-2u_{i,j}+u_{i-1,j})+(u_{i,j-1}-2u_{i,j-1}+u_{i-1,j-1})\right] \qquad (4.48)$$

The formulae (4..47) and (4.48) hold good for all values of $\frac{k}{h}$. The use of formula (4.46) is demonstrated in the following example.

4.7.4 Stability Analysis

In order to discuss stability, the difference equation (4.46) can be written as

$$u_m^{n+1}=2(1-\alpha^2)\,u_m^n+\alpha^2(u_{m+1}^n+u_{m-1}^n)-u_m^{n-1} \qquad (4.49)$$

Substituting $$u_m^n=A\xi^4 e^{i\beta mh} \qquad (4.50)$$

In equation (4.49) and simplifying [27], we get;

$$\xi^2-(2-4\alpha^2\sin^2\phi)\xi+1=0 \qquad (4.51)$$

Where $\mathrm{j}=\beta h/2$. the roots of equation (4.51) are given by

$$\xi_{1,2}=(1-2\alpha^2\sin^2\phi)\pm\sqrt{(1-2\alpha^2\sin^2\phi)^2-1} \qquad (4.52)$$

Now the following possibilities arise:

- if $\left|1-2\alpha^2\sin^2\phi\right|>1$, then $|\xi_1|>1$ and hence the scheme is unstable.
- if $\left|1-2\alpha^2\sin^2\phi\right|<1$, then $\xi_{1,2}$ are in a complex pair with magnitude $|\xi|=1.$
- If $\left|1-2\alpha^2\sin^2\phi\right|=1$, then $|\xi_{1,2}|=1.$

Thus, the scheme (4.49) is stable for

$$-1 \le 1 - 2\alpha^2 \sin^2 \phi \le 1$$

Which gives us the condition $\alpha \le 1$

Example (4.6)

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 16 \frac{\partial^2 u}{\partial x^2}, \text{ for } 0 < x < 5 \text{ and } 0 < t < 1.25$$

taking $\Delta x = 1$, and subject to the boundary and initial conditions

$u(0,t) = u(5,t) = 0$, $\frac{\partial u}{\partial t}(x,0) = 0$, and $u(x,0) = x^2(5-x)$

Solution:

Here $c^2 = 16$, therefore, the difference equation for the given equation is

$$u_{i,j+1} = 2(1-16\alpha^2)\, u_{i,j} + 16\alpha^2 (u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} \qquad (4.53)$$

Where $\alpha = \frac{k}{h}$. Taking $h = 1$ and choosing k so that the coefficient of $u_{i,j}$ vanishes, we have

$$16\alpha^2 = 1 \Rightarrow \frac{k^2}{h^2} = \frac{1}{16} \Rightarrow k = \frac{h}{4} = \frac{1}{4}$$

Therefore (4.53) reduces to

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \qquad (4.54)$$

Which gives us a convergent solution as $\frac{k}{h}<1$. Now since $u(0,t)=u(5,t)=0$, therefore $u_{0,j}=0$ and $u_{5,j}=0$, for all j.

Also $u(x,0)=x^2(5-x)$

$\Rightarrow$ $u_{i,0}=i^2(5-i)=4,12,18,16$, for $i=1,2,3,4,$ at $t=0$

Finally, since $\frac{\partial u}{\partial t}(x,0)=0,$ therefore we have

$$\frac{u_{i,j+1}-u_{i,j}}{k}=0 \quad \text{for } j=0 \text{ i.e } u_{i,1}=u_{i,0}$$

Which means the entries of the second row are the same of those of the first row. Putting $j=1$ in (4.54), we get

$$u_{i,2}=u_{i-1,1}+u_{i+1,1}-u_{i,0}$$

Now taking $i=1,2,3,4$ successively, we obtain

$$u_{1,2}=0+12-4=8$$

$$u_{2,2}=4+18-12=10$$

$$u_{3,2}=12+16-18=10$$

$$u_{4,2}=18+0-16=2$$

Putting $j=2,3,4$ successively in (4.54) the entries of the remaining forth, fifth and sixth row and the obtained values of the $u_{i,j}$ are given in the following table [27],

Table 4.4 : Stability and Convergence

i/j	0	1	2	3	4	5
0	0	4	12	18	16	0
1	0	4	12	18	16	0
2	0	8	10	10	2	0
3	0	6	6	-6	-6	0
4	0	-2	-10	-10	-8	0

4.7.5 The Local Truncation Error and Consistence of The Finite Difference Schemes

The local truncation error is defined as the difference of the differential equation and the finite difference scheme. The finite difference scheme is called *consistent* if the limit of the local truncation error is zero [36] as h and/ or Δt approach zero.

The local truncation errors for the forward, backward, and central finite difference are

$$T_h(D_+) = f'(x) - \frac{f(x+h) - f(x)}{h} = \frac{h}{2} f''(\xi)$$

$$T_h(D_-) = f'(x) - \frac{f(x) - f(x-h)}{h} = -\frac{h}{2} f''(\xi)$$

$$T_h(D_0) = f'(x) - \frac{f(x+h) - f(x-h)}{2h} = \frac{h^2}{6} f^{(4)}(\xi).$$

In all three cases, we have $\lim_{h \to 0} T_h = 0$. Therefore they are all consistent.

4.7.6 The Truncation Error of The Finite Difference Method for The 1D One-Way Wave Equation

The differential equation of the one way wave equation is

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=f\ (x,t),\qquad a<x<b,\quad t>0.$$

A simple finite difference scheme [36] is

$$\frac{u_i^{k+1}-u_i^k}{\Delta t}+c\frac{u_i^k-u_{i-1}^k}{\Delta x}=f_i^{\,k},\qquad i=0,1,\ldots,\quad \text{if } c>0.$$

$$T_h=\frac{u(x,t+h)-u(x,t)}{\Delta t}+c\frac{u(x+h,t)-u(x,t)}{h}-f\ (x,t),\ \ c>0$$

$$=u_t(x,t)+\frac{\Delta t}{2}u_{tt}(x,\eta)-cu_x(x,t)-c\ \frac{h^2}{2}u_{xx}(\xi,t)$$

$$=O(\Delta t,h),\quad \lim_{\Delta t\to 0,h\to o}T_h=0.$$

Therefore the finite difference scheme is consistent and is first order in time and first order in space.

4.7.7 The Stability of Finite Difference Scheme

As we see from the numerical test, whether a finite difference scheme can work depend on the choice of Δt even if the finite difference scheme is consistent. The stability condition is a requirement that the error in the computed solution would be amplified in the subsequent computations [36]. An intuitive definition is that

$$\left|u_1^{k+1}\right| + \left|u_2^{k+1}\right| + \left|u_3^{k+1}\right| + \ldots + \left|u_{m-2}^{k+1}\right| + \left|u_{m-1}^{k+1}\right| + \left|u_m^{k+1}\right| \leq$$

$$\left|u_1^{k}\right| + \left|u_2^{k}\right| + \left|u_3^{k}\right| + \ldots + \left|u_{m-2}^{k}\right| + \left|u_{m-1}^{k}\right| + \left|u_m^{k}\right|$$

The global error (overall error) of a finite difference scheme is the absolute error of the computed solution. That is

$$E = u^{k_{final}} - u^{(k_{final}\Delta t)}$$

For one way wave equation, the global error is

$$E_i = u_i^{k_{final}} - u(x_i, t_{final}),\ i = 1,2,\ldots,m$$

Usually we use one measurement called the infinity normal of the error

$$\|E\|_\infty = \max\left\{|E_1|,|E_2|,\ldots,|E_{m-1}|,|E_m|\right\}$$

A finite difference method is *convergent* to the true solution if the global error approaches to zero as Δ and h approach to zero.

***Theorem* 4.7.1:** *A consistent and stable finite difference method is convergent.*

Conclusion

In this thesis we have considered Maxwell's equations due to their wide range of applications. These equations synthesized all electromgnetic phenomena into four equations.

Maxwell's first equation is Gauss's law of electric field, the second equation is Gauss's law of magnetic field, the third equation is Faraday's law and the fourth equation is an extension of Ampere's law. All these equations have been derived in various forms using the basic properties of the electromagnetic theory. Though it is well known that Maxwell's equations are hard to solve analytically, however, we were able to use some well known analytical methods namely, separation of variables, eigen-function expansion method, integral methods, conformal mapping and Green's functions to solve these equations for some special cases.

For the numerical treatment of the Maxwell's equation, we have used the Finite Difference (FD) and the Finite Difference Time Domain (FDTDM) methods. The FDTDM is based on the Yee Algorithm. This Algorithm proposed a discrete solution to Maxwell's equations based on central difference approximations of the spatial and temporal derivatives of the curl equations. Moreover, the convergence, stability and error analysis for these numerical methods have been thoroughly investigated.

We have concluded that the FDTDM implementing the Yee Algorithm has proved to be a very reliable and efficient numerical method for solving the Maxwell's equations.

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جامعة النجاح الوطنية
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قدمت هذه الاطروحة استكمالا لمتطلبات درجة الماجستير في الرياضيات المحوسبة بكلية الدراسات العليا في جامعة النجاح الوطنية في نابلس، فلسطين .

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الملخص

قدمت هذه الرسالة ملخص لمعادلات ماكسويل والتي هي عبارة عن مجموعة من أربع معادلات تصف سلوك وتغيرات الحقلين الكهربائي والمغناطيسي وتأثراتهما مع المادة وتحولاتهما إلى أشكال أخرى من الطاقة. هذه القوانين من وضع الفيزيائي جيمس ماكسويل . وهذه المعادلات تصف العلاقات المتبادلة بين كل من المجالات الكهربائية والمجالات المغناطيسية والشحنات الكهربائية والتيار الكهربائي. وحيث تم في البداية العمل على اشتقاق لهذه المعادلات بالرجوع الى المتغيرات والكميات الفيزيائية المرتبطة بها. وتم الحصول على الشكلين المتعارف عليهما لمعادلات ماكسويل وهما الشكل التفاضلي النقطي والشكل التكاملي. كما وتم عرض بعضا من الحلول التحليلية المختلفة لهذه المعادلات والتي تضمنت على سبيل المثال لا الحصر: طريقة فصل المتغيرات، تحويلات لابلاس وفورير، وغيرها . كما وناقشت الرسالة بعضا من الطرق العددية في حل معادلات ماكسويل وهذه الطريقة هي الفروقات المنتهية حيث تم عرض حلول لبعض الأمثلة عليها و كذلك دراسة لبعض الأخطاء وحالة الثبات لهذه الطريقة .