An-Najah National University

Faculty of Graduate Studies

Wavelets Numerical Methods for Solving Differential Equations

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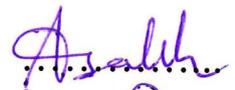
III
Dedication

I dedicate this thesis

To my beloved Palestine

To my parents

To my sisters

To my family

To my friends

Who helped me, stood by me and encouraged me.

Acknowledgement

All praise be to almighty God, without whose mercy and clemency nothing would have been possible. I wish to express my appreciation to Dr. Anwar Saleh, my advisor, for giving me all the necessary support I needed to complete this work, without him this work would not have been accomplished.

I would sincerely like to thank and deeply grateful to my external examiner Dr. Abdel Halim Ziqan and to internal examiner Dr. Samir Matar for their useful and valuable comments. Also, my great thanks are due to my family for their support, encouragement and great efforts for me.

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V

الإقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان:

Wavelets Numerical Methods for Solving Differential Equations

أقر بأن ما اشتملت عليه هذه الرسالة إنما هي نتاج جهدي الخاص، باستثناء ما تمت الإشارة إليه حيثما ورد، وان هذه الرسالة ككل، أو أي جزء منها لم يقدم من قبل لنيل أي درجه علميه أو بحثي لدى أية مِؤسسه تعليمية أو بحثيه أخرى.

Declaration

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

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Abstract

In this thesis, a computational study of the relatively new numerical methods of Haar wavelets for solving linear differential equations is used. A comparison between the new method and some classical methods for linear differential equations has been made. The aim is to show the efficiency of the presented method and its advantage over other method. The new method is simple and its numerical results are close or more accurate than some classical methods.

Chapter One

Introduction

1.1. Overview

Wavelet analysis is an exciting new method for solving difficult problems in mathematics, physics, and engineering [7]. Wavelets are functions that satisfy certain mathematical requirement and can be used to represent a function, such as the solution of ODE and PDE. Representation of functions is an old subject.

Weierstrass theorem [3] guarantees the existence of polynomial that approximates any continuous function on any closed interval of $\mathbb{R}$ to any level. In 1714 Taylor theorem [1,3] represents the polynomial that approximates a function using the set $\{(x - x_0)^n\}_{n=0}^{\infty}$ as basis. In 1808, Joseph Fourier [6] used the set $\{sin(nx), cos(nx)\}_{n=1}^{\infty}$ as basis to represent functions of period $p = 2\pi$, then generalized to functions of any period $p = 2L$. Wavelet representation of functions uses the wavelet basis defined in chapter three.

The wavelet decomposition analysis is used most often in wavelet signal processing [7]. It is used in signal compression as well as in signal identification. Wavelet transform of a function, as Fourier transform, is powerful tool for analyzing the components of stationary phenomena. However, the wavelet transform has the advantage of the ability of analyzing nonstationary phenomena where Fourier transform fails [7,8,28]. Wavelet transform, as one of the mathematical real or complex valued function, is one which has become widely used in various fields of

application, mainly medicine, communication, computer software and human related applications. In specific, wavelet can be found in scanning, disease diagnosis to help doctors do their job precisely in this human sensitive field. It can also help encode audio and video signals in the field of telecommunications. As well, there are other useful applications which can effectively help intelligence agencies recognize the tinniest details of human bodies for security purposes and in cases of terrorist acts, airplane collapse, ship wreckage, and other human verification uses. For example, the Federal Bureau of Investigation (FBI) of the United States uses wavelet application to identify and verify millions of people's fingerprints. In the future, it is expected that mathematical wavelet technology will cover hundreds of applications and it will mainly focus human welfare and healthcare subjects to achieve the best results possible [7,8,9,30].

In chapter one, we give a brief history of wavelets and a review of literature on subject of wavelets. In chapter two, we review some classical numerical methods for ordinary and partial differential equations, for comparison with the Haar wavelet method used in this thesis. In chapters three wavelets and their applications are considered. In chapter four, the Haar wavelet method for solving differential equations is given. In chapter five, numerical examples of the Haar wavelet method are given and the results are compared with results from the classical methods introduced in chapter two. Conclusion is given in section 5.4

1.2 A brief history of the development of wavelets in solving differential equations

The word "wavelet" is due to Morlet and Grossmann. In the early 1980s, influenced by ideas from both pure and applied mathematics, they used the French word "ondelette", meaning "small wave". Soon it was transferred to English by translating "onde" into "wave", giving the name "wavelet"[7]. Wavelets were developed independently in the fields of mathematics, quantum physics, electrical engineering, and medical technology etc.

The main branch of mathematics leading to wavelets began with Joseph Fourier in 1807 with his theories of frequency analysis. Fourier's efforts with frequency analysis lead to the well-known Fourier analysis. Fourier's work is based on the fact that periodic functions can be represented as a linear combination of sines and cosines [6]. Another contribution of Fourier is the Fourier transform. It transforms a function f from its time domain into its frequency domain [28].

The next known link to wavelet is due to Alfred Haar in 1910. It appeared in the appendix of a thesis he had written to obtain his doctoral degree. Haar contribution to wavelets is very evident. The well-known Haar wavelet is the simplest and the oldest of the wavelet family. In the year 1910, Alfred Haar introduced a function, which presents a rectangular pulse pair. Then various generalizations were proposed [9], see figure 1.1
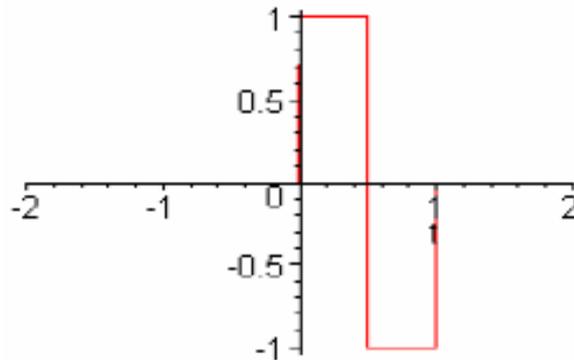
Figure 1.1: Haar wavelet

After Haar contribution to wavelets there was once a gain a gap of time in research about the functions until 1954 when Paul Levy used wavelets in his research on Brownian motion [7]. He discovered that the scale-varying basis function-created by Haar (i.e. Haar wavelets) were a better basis than the Fourier basis functions. Unlike the Haar basis function, which can be chopped up into different intervals – such as the interval from 0 to 1 or the interval from 0 to $\frac{1}{2}$ and $\frac{1}{2}$ to 1, the Fourier basis functions have only one interval. Therefore, the Haar wavelets can be much more precise in modeling a function.

In the year 1975, Jean Morlet was the first researcher to use the term "wavelet" to describe his functions [8]. More specifically, they were called "wavelets of constant slope".

In 1980s, it Ingrid Daubechies generalized of Haar wavelet. In fact, Haar wavelet is the Daubechies wavelet of order one. Wavelet methods have been applied for solving differential equations from the beginning of the early 1990s [9], see figure 1.2
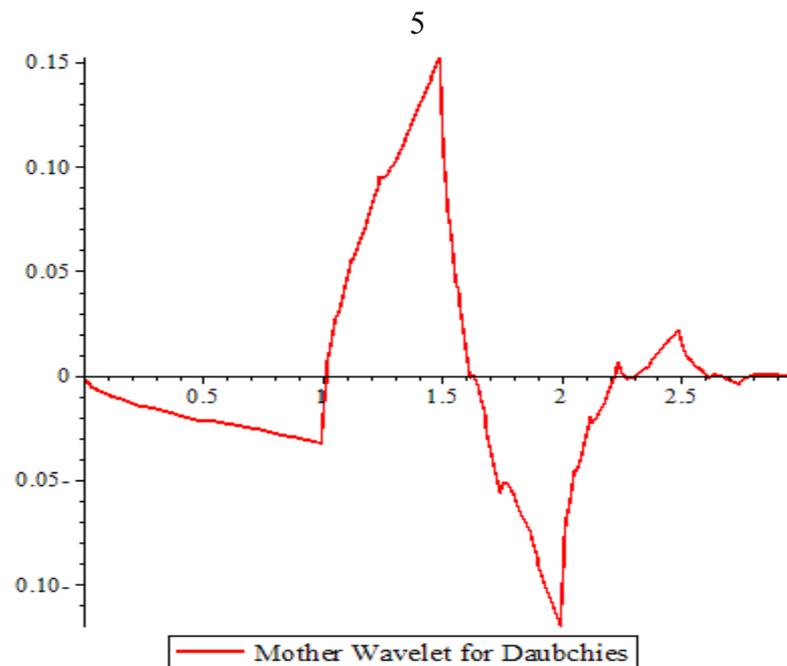
Figure 1.2: Mother wavelet for Dauchies.

In the year 1997, Chen and Hsiao introduced a wavelet method for solving differential equations. The method is based on expanding the highest derivative in the differential equation in terms of Haar wavelets then integrating the expansion to get the solution. To simplify things, they introduce the Haar operational matrix for the integrals of the Haar function vector and put the application for the Haar analysis into the dynamic system [11, 13]. Phang Chang and Phang Piau [11] also gave a simple matrix method to solve ordinary differential equations. From 2005 to 2014, ÜloLepik [15, 16] presented an application of the Haar wavelets for solution of linear integral equations and numerical solution of differential equations using Haar wavelets, then he presented Haar wavelet methods for solving evolution equations and Haar wavelet methods for nonlinear integro-differential equations. Lepik presented application of nonuniform Haar wavelets for solving integral and differential equations. A year later

he presented Haar wavelet method for solving higher order differential equations and integral equations. In the years from 2009 to 2014, Hariharan et al[13,17] established the solution of several differential equations in applications. Among them convection-diffusion equation, finite length beams equation, linear and nonlinear Klein- Gorden equation, Sine Gorden equation and some nonlinear parabolic equations by the Haar wavelet method. In the year 2007 Castro[16] presented a wavelet collocation method for the static analysis of sandwich plates using a layer wise theory. In the year 2010 Fazal-I-Haq [21] introduced a Haar wavelet method for solving eight-order boundary value problems. From 2010-2013, Siraj-ul-Islam et al[22,24] presented numerical solution of second-order boundary-value problems by the Haar wavelets, Haar wavelet collocation method for numerical solution of boundary layer fluid flow problems. A year later they presented a numerical assessment of parabolic partial differential equations using Haar and Legendre wavelets. In the year 2011, Zhi Shi and Yong-Yan Cao [11, 12] established the Haar wavelet method for solving higher order differential equations, a year later they presented poisson equations and biharmonic equations on a rectangle. In the year 2012, Naresh Berwal [19] presented the Haar wavelet method for solving wave-like equation. In the year 2013 Imran Aziz [23] presented a wavelets collocation methods for the a numerical solution of elliptic BVPs. It is obvious that the subject of wavelets methods for solving differential equations has attracted many researchers in the last 10 years. And that is due to its potential in solving many problems.

Chapter Two

Overview of numerical methods for differential equations

A differential equation relates an unknown function to its derivatives. In applications, functions usually represent physical quantities and the derivatives represent their rates of change. Differential equations play an important role in modeling of physical problems in science, engineering and economics. Differential equations describe a wide range of natural phenomena, such as sound, heat, electrostatics, electrodynamics, fluid flow, elasticity, and quantum mechanics. These seemingly distinct physical phenomena can be formalized similarly in terms of PDEs.

While, analytical methods can be used to solve some differential equations, many, if not most differential equations, can't be solved analytically. Numerical methods for differential equations are considered when the differential equations, ordinary or partial, can't be solved directly by analytical methods.

Among the classical numerical methods for solving ordinary differential equations are; Euler method, higher-order Taylor methods, Runge-Kutta methods, the multistep methods, linear shooting method and finite difference methods. Among the classical numerical methods for solving partial differential equations are; finite difference methods, finite element method, Adomain Decomposition Method (ADM), and multigrid methods.

2.1 Numerical methods for solving ordinary differential equations

In this section, we give an overview of some classical numerical methods for ordinary differential equations. we will focus on the following methods

for initial value problems: Euler methods, higher-order Taylor methods, and Runge-Kutta methods. We will also focus on the following methods for boundary value problems: The linear shooting methods, finite difference methods.

2.1.1 Numerical methods for initial-value problems

Consider the first-order initial value problem

$$\frac{du}{dx} = f(x, u), \quad for \ a \leq x \leq b, \quad u(a) = \alpha \tag{2.1}$$

In the following subsections, we overview three classical numerical methods for this IVP.

2.1.1.1 Euler's method

Euler's method was established by the Swiss mathematician and physicist Leonhard Euler, who treated it in his book "institutionum calculi integral" which is published between 1768 and 1770. He made important discoveries in fields as diverse as infinitesimal calculus and graph theory. He also introduced much of the modern mathematical terminology.

Euler's method is considered the most elementary approximation technique for solving IVPs. The object of Euler's method is to approximate the solution to the well-posed initial-value problem (2.1) [26,29].

Once the approximate solution is obtained at given points, the approximate solution at other points in the interval can be found by interpolation.

The method starts by dividing the interval $[a, b]$ into some N subintervals with

$$x_i = a + ih, \qquad for \ each \ i = 0,1,2, \dots, N \qquad\qquad (2.2)$$

$$where \ \ h = \frac{b-a}{N} = x_{i+1} - x_i \qquad\qquad (2.3)$$

Using first-order forward-difference formula for u', we get

$$\frac{u(x_{i+1}) - u(x_i)}{h} - \frac{h}{2}u''(\xi) = f(x_i, u(x_i)) \qquad\qquad (2.4)$$

Drop the error term and let $u_i = u(x_i)$

$$u_0 = \alpha$$

$$u_{i+1} = u_i + hf(x_i, u_i), \ \ i = 0,1, \dots, N-1 \qquad\qquad (2.5)$$

This is Euler's method for the IVP in (2.1)

Example 2.1: We use Euler's method to approximate the solution to the IVP

$$u' = \frac{sin(2x) - 2xu}{x^2}, \quad 1 \le x \le 2, \quad u(1) = 2$$

With exact solution

$$u(x) = \frac{4 + cos(2) - cos(2x)}{2x^2}$$

with $N = 10$ subintervals

$$h = \frac{1}{10}$$

$$f(x,u) = \frac{sin(2x) - 2xu}{x^2}$$

$$u_0 = u(1) = 2,$$

$$u_{i+1} = u_i + hf(x_i, u_i), \text{ for each } i = 1,2, \dots, 9$$

 The results of Euler's method are given in table 2.1

Table 2.1: Maximum absolute error of Euler's method for IVP is 0.062594347.

i	x_i	u_i
0	1.0	2.0000000
1	1.1	1.6909297
2	1.2	1.4503059
3	1.3	1.2554954
4	1.4	1.0928453
5	1.5	0.9538157
6	1.6	0.8329123
7	1.7	0.7265181
8	1.8	0.6322081
9	1.9	0.5483003
10	2.0	0.4736355

The results are obtained using Maple software.

2.1.1.2 Higher-order Taylor methods for solving initial-value problems

The error in Euler's method is of order one. For more accurate approximations, we need methods of higher order errors. To do so, we use more terms in Taylor theorem to approximate the derivative in an IVP and the result is a higher order Taylor method. Euler's method is a first order Taylor method [26,29].

The nth-order Taylor method for the IVP 2.1 is given as

$$u_0 = \alpha$$

$$u_{i+1} = u_i + hT^{(n)}(x_i, u_i), \quad each \ i = 0,1, \dots, N-1 \tag{2.6}$$

where

$$T^{(n)}(x_i, u_i) = f(x_i, u_i) + \frac{h}{2}f'(x_i, u_i) + \cdots + \frac{h^{n-1}}{n!}f^{(n-1)}(x_i, u_i) \qquad (2.7)$$

Example 2.2: Consider the IVP given in example 2.1

Table 2.2 shows the approximation of solution to the IVP in example 2.1 using first-order and fourth-order Taylor methods with $N = 10$ subintervals.

The results of Taylor methods of order one and four are given in table 2.2

Table 2.2: Maximum absolute error of Taylor of order one for IVP is 0.062594347

i	x_i	$u_i \; of \; order \; one$	$u_i \; of \; order \; four$
0	1.0	2.0000000	2.0000000
1	1.1	1.6909297	1.7241829
2	1.2	1.4503059	1.5005274
3	1.3	1.2554954	1.3139293
4	1.4	1.0928453	1.1547091
5	1.5	0.9538157	1.0165024
6	1.6	0.8329123	0.8950361
7	1.7	0.7265181	0.7873883
8	1.8	0.6322081	0.6915241
9	1.9	0.5483003	0.6059965
10	2.0	0.4736355	0.5297472

Maximum absolute error of Taylor of order four for IVP is 0.0001003

2.1.1.3 Runge-Kutta methods for solving initial-value problems

The computation of derivatives in higher-order Taylor methods becomes complicated as the order increases. Runge-kutta methods avoid the computation of such derivatives and only functional evaluations are used [26,29].

Runge-Kutta method of order two or the midpoint method.

$$u_0 = \alpha$$

$$k_1 = hf(x_i, u_i),$$

$$k_2 = hf\left(x_i + \frac{h}{2}, u_i + \frac{1}{2}k_1\right),$$

$$u_{i+1} = u_i + k_2 \tag{2.8}$$

$$for\ each\ i = 0,1,2, \dots, N - 1$$

Runge-Kutta method of order four (RK4)

$$u_0 = \alpha$$

$$k_1 = hf(x_i, u_i), \tag{2.9}$$

$$k_2 = hf\left(x_i + \frac{h}{2}, u_i + \frac{1}{2}k_1\right), \tag{2.10}$$

$$k_3 = hf\left(x_i + \frac{h}{2}, u_i + \frac{1}{2}k_2\right), \tag{2.11}$$

$$k_4 = hf(x_{i+1}, u_i + k_3), \tag{2.12}$$

$$u_{i+1} = u_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{2.13}$$

Example 2.3: Consider the IVP in example 2.1

$$u' = \frac{sin(2x) - 2xu}{x^2}, \quad 1 \le x \le 2, \quad u(1) = 2$$

The solution by the midpoint method "RK2" and Runge-Kutta method order four "RK4" using $N = 10$ are given in table 2.3

Table 2.3: The results of RK2 and RK4

i	x_i	$RK2$	$RK4$
0	1.0	2.0000000	2.0000000
1	1.1	1.7251529	1.7241157
2	1.2	1.5019911	1.5004362
3	1.3	1.3156406	1.3138325
4	1.4	1.1565391	1.1546145
5	1.5	1.0183834	1.0164134
6	1.6	0.8969326	0.8949538
7	1.7	0.7892816	0.7873127
8	1.8	0.6934041	0.6914550
9	1.9	0.6078570	0.6099333
10	2.0	0.5315839	0.5296893

Maximum absolute error of RK2 for IVP is 0.19818485×10^{-2}

Maximum absolute error of RK4 for IVP is 0.34785891×10^{-5}

2.1.2 Numerical methods for boundary-value problems

A Boundary Value Problem (BVP) is a differential equation together with a set of boundary conditions. A solution to a BVP is a solution to the differential equation which also satisfies the boundary conditions. We will focus on two methods to approximate the solution of a second order of linear BVP. The first method is the linear shooting method and the second method is the linear finite difference method. We will focus on linear equations since they will be used for comparison with wavelet methods in chapter five.

A second–order linear BVP can be written in the form:

$$u'' = p(x)u' + q(x)u + r(x) \text{ for } a \leq x \leq b \qquad (2.14)$$

$u(a) = \alpha$ and $u(b) = \beta$

2.1.2.1 The Linear shooting method

The linear shooting method for linear BVPs is based on the replacement of the linear BVP by two IVPs. Then the solution of the BVP is a linear combinations of the solutions of the two IVPs[26,32].

$$u_1'' = p(x)u_1' + q(x)u_1 + r(x), \quad for \ a \le x \le b, \tag{2.15}$$

with $u_1(a) = \alpha$ and $u_1'(a) = 0$

and

$$u_2'' = p(x)u_2' + q(x)u_2, \quad for \ a \le x \le b, \tag{2.16}$$

with $u_2(a) = 0$ and $u_2'(a) = \beta$

The solution to the BVP (2.14) is

$$u(x_i) = u_1(x_i) + \frac{\beta - u_1(b)}{u_2(b)} u_2(x_i) \tag{2.17}$$

Where u_1 and u_2 are the solutions of (2.15) and (2.16) respectivly

Example 2.4: Consider the BVP:

$$u'' = -3u' + 2u + 2x + 3, \quad 0 \le x \le 1, \qquad u(0) = 2 \text{ and } u(1) = 1$$

Using the linear shooting method with $N = 10$

First, we convert this problem into two IVPs

$$u_1'' = -3u_1' + 2u_1 + 2x + 3, \quad 0 \le x \le 1, \quad u_1(0) = 2 \text{ and } u_1(1) = 0;$$

and

$$u_2'' = -3u_2' + 2u_2 + 2x, \quad 0 \le x \le 1, \quad u_2(0) = 0 \text{ and } u_2(1) = 1;$$

The solution to the BVP is

$$u(x_i) = u_1(x_i) + \frac{1 - u_1(1)}{u_2(1)} u_2(x_i)$$

The discrete solution of the two IVPs together with the BVP is given in table 2.4

Table 2.4: The results of linear shooting

i	x_i	$u_1(x_i)$	$u_2(x_i)$	$u(x_i)$
0	0.0	2.00000000	-7.19616313	2.00000000
1	0.1	1.40843171	-4.77470838	1.40843171
2	0.2	1.02226375	-3.04605634	1.02226375
3	0.3	0.78331776	-1.80076394	0.78331776
4	0.4	0.65103904	-0.89202648	0.65103904
5	0.5	0.59722789	-0.21690543	0.59722789
6	0.6	0.60234998	-0.29682014	0.60234998
7	0.7	0.65295283	0.69986330	0.65295283
8	0.8	0.73985699	1.02788028	0.73985699
9	0.9	0.85688993	1.30598925	0.85688993
10	1.0	1.00000000	1.55193574	1.00000000

2.1.2.2 Finite difference methods for linear boundary-value problems

In this section, we review the linear finite difference method for linear

BVPs. We will use this method in chapter five for comparison with

wavelets methods.

Main idea of finite difference methods.

Divide the domain into a member of subintervals, then approximate the

derivatives using numerical differentiation formulas at each interior point.

The result is a linear system of equations. This process is called

discretization of the continuous problem [26, 29]. The solution of the linear system is the discrete solution of the linear BVP.

Example 2.5: Consider the BVP

$$u'' = 4(u - x), \quad 0 \le x \le 1, \qquad u(0) = 0 \ and \ u(1) = 2.$$

use linear finite difference method to approximate the solution

With $N = 9$, then $h = \frac{b-a}{N+1} = \frac{1}{10}$, ($x_0 \ and \ x_{10}$ are boundary points)

The mesh points are

$x_0 \quad x_1 \quad x_2 \quad x_3 \ x_4 \ x_5 \quad x_6 \ x_7 \ x_8 \ x_9 \quad x_{10}$

0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0

Use second-order centered-difference formula for $u''(x_i), i = 1,2,\dots,9$ and drop the error term, we get

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = 4u_i - 4x_i$$

$$100(u_{i+1} - 2u_i + u_{i-1}) = 400u_i - 400x_i, \quad for \ i = 1,2,\dots,9$$

$$u_0 = 0, \ u_{10} = 2$$

Solve the system to get the discrete solution. The results are given in table 2.5

Table 2.5: The result of linear finite difference method.

i	x_i	u_i
0	0.0	0.00000000
1	0.1	0.10882378
2	0.2	0.22205946
3	0.3	0.34191297
4	0.4	0.47169324
5	0.5	0.61636364
6	0.6	0.78336924
7	0.7	0.98387764
8	0.8	1.23464024
9	0.9	1.56078414
10	1.0	2.00000000

2.2 Finite difference methods for solving partial differential equations

Just like finite difference methods for BVPs, finite difference methods for PDEs, replace the derivatives by numerical differentiation formals. However, for PDEs in two dimensions (two independent variables), the domain is a plane region. To discretize the problem, generate a grid and approximate the derivatives at each interior grid point to get a linear system [29].

Example 2.6: Consider the wave like problem.

$$u_{tt} - \frac{x^2}{2} u_{xx} = 0, \quad 0 < x < 1, \quad 0 < t < 0.01$$

$$u(0, t) = 0, \quad u(1, t) = 1 + sinh(t)$$

$$u(x, 0) = x, \quad u_t(x, 0) = x^2$$

Using finite difference method with $N + 1 = 32$ subinterval of $[0,1]$

And $M = 20$ subinterval of $[0,0.01]$

The space step size is $h = \dfrac{1}{32}$

The time step size is $k = \dfrac{1}{20}$

$x_i = ih, \qquad i = 0,1,2,\dots,N. \qquad t_j = jk, \quad j = 0,1,2,\dots,M-1$

Let u_{ij} be the approximation of the solution $u(x_i,t_j)$,

$$u_{tt}(x_i,t_j) = \frac{x_i^2}{2}\, u_{xx}(x_i,t_j)$$

Using 2nd $-$order centered-difference formula for u_{tt} and u_{xx} at the grid point (x_i,t_j) and dropping the error terms, we get

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = \frac{x_i^2}{2}\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = \frac{k^2}{2h^2}x_i^2[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}], \ \ \text{let } \lambda = \frac{k^2}{2h^2}$$

$$u_{i,j+1} = \lambda x_i^2 u_{i+1,j} + 2(1 - \lambda x_i^2)u_{i,j} - \lambda x_i^2 u_{i-1,j} - u_{i,j-1}$$

So when $j+1$ is replaced $j+2$, we get

$$u_{i,j+2} = \lambda x_i^2 u_{i+1,j+1} + 2(1 - \lambda x_i^2)u_{i,j+1} - \lambda x_i^2 u_{i-1,j+1} - u_{i,j}$$

For $i = 1,2,3,4,\dots,32$ and $j = 1,2,3,4,\dots,20$.

The boundary conditions imply: $u_{0j} = 0$ and $u_{N+1,j} = 1 + sinh(jk)$

Initial conditions imply: $u_{i0} = ih$ and $u_{iN+1} = (ih)^2$

Table 2.6: The result of finite difference method for wave-like equation

i	x_i	u_i
1	0.03125	0.0312597648
3	0.09375	0.0938378934
5	0.15625	0.1564941463
7	0.21875	0.2192285265
9	0.28125	0.2820410392
11	0.34375	0.3449316610
13	0.40625	0.4079004237
15	0.46875	0.4709473089
17	0.53125	0.5340723150
19	0.59375	0.5972754320
21	0.65625	0.6605567180
23	0.71875	0.7239161120
25	0.78125	0.7873536280
27	0.84375	0.8508692660
29	0.90625	0.9144630310
31	0.96875	0.9781349360

Chapter Three

Wavelets analysis

Wavelets can be used as basis for expanding a large class of functions. Wavelet means a small wave where the sinusoids used in Fourier series are big waves. In brief, a wavelet is an oscillation that decays quickly. Fourier series uses the trigonometric functions; sine and cosine that do not decay, i.e. they have infinite support. In the wavelet expansion of a function, the bases are wavelets that decay quickly, i.e. they have finite support [8].

3.1 Wavelet transforms (WT)

In the year 1982, Jean Morlet introduced the idea of the wavelet transform and provided a new mathematical tool for wavelet analysis. **Definition 3.1[8]: Mother wavelets**

A mother wavelet is a function $\psi(x)$ that satisfies the following conditions

$$1. \quad c_\psi = \int_0^\infty \frac{|\Psi(w)|^2}{|w|} dw \ < \infty \qquad (3.1)$$

where $\Psi(w)$ is the Fourier transform of $\psi(x)$ as a result [4,6]

$$\int_{-\infty}^{\infty} \psi(x)\, dx = 0 \qquad (3.2)$$

2. a wavelet function is unit energy; that is

$$\int_{-\infty}^{\infty} |\psi(x)|^2\, dx = 1 \qquad (3.3)$$

Where the energy of a function is defined to be the squared function integrated over its domain by satisfying both equation (3.1, 3.2), the wavelet function must have nonzero entries, but all departures from zero must cancel out.

3. ψ is smooth, may be even infinitely often differentiable [30].

4. ψ has a computationally convenient form; for example that ψ is a piecewise polynomial; i.e. as plane.

5. ψ has compact support; i.e. all function values are zero outside a certain bounded interval.

A wavelet $\psi_{a,b}(x)$ is constructed from a mother wavelet by scaling and translation.

$$\psi_{a,b}(x) = \frac{1}{\sqrt{|a|}} \psi(\frac{x-b}{a}) , \ a, b \in \mathbb{R} \ , a \neq 0 \tag{3.4}$$

The parameter a is the scaling parameter or scale, and it measures the degree of compression. The parameter b is the translation parameter which determines the $x -$ location of the wavelet.

If $|a| < 1$, then the wavelet is in the compressed version (smaller support in the $x -$ domain) of the mother wavelet and corresponds mainly to higher frequencies. On the other hand, when $|a| > 1$, then $\psi_{a,b}(x)$ has a larger $x -$ width than $\psi(x)$ and corresponds to two lower frequencies, thus wavelets have space widths adapted to their frequency. This is the main reason for the success of the Jean Morlet wavelets in signal processing and time frequency signal analysis.

There are three types of wavelet transform: The continuous wavelet transform (CWT), the discrete wavelet transform (DWT), and the wavelet series. The CWT transforms a continuous function to a continuous function, The DWT transforms a discrete function to discrete function, and the wavelet series transforms a continuous function to discrete function.

3.1.1 The continuous wavelet transforms and it's inverse

Definition 3.2[5]: The integral wavelet transform or continuous wavelet transform of a function $f(x) \in L^2$ with respect to some mother wavelet ψ is defined as

$$W_\psi f(a, b) = \int_{-\infty}^{\infty} f(x)\psi_{a,b}dx \tag{3.5}$$

where

$$\psi_{a,b}(x) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{x-b}{a}\right), \quad a \neq 0 \tag{3.6}$$

Definition 3.3[6]: The factor $\frac{1}{\sqrt{a}} = a^{-\frac{1}{2}}$ is a normalizing factor. As a result

$$\left\|\psi_{a,b}\right\|_2 = \|\psi\|_2 \tag{3.7}$$

$$\int_{-\infty}^{\infty} \left|\psi_{a,b}(x)\right|^2 dx = \int_{-\infty}^{\infty} |\psi(x)|^2 dx \tag{3.8}$$

Example 3.1: Consider the Mexican hat wavelet

$$\psi(x) = (1 - 2x^2)e^{-x^2}$$

also

$$\psi_{a,b}(x) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{x-b}{a}\right) = \frac{1}{\sqrt{a}}(1 - 2\left(\frac{x-b}{a}\right)^2)e^{-(\frac{x-b}{a})^2}$$

Fixing $a = 1$ and the translating factor b to any integer,

$$\psi_{1,b} = \psi(x-b) = (1-2(x-b)^2)e^{-(x-b)^2} \text{ (see Figuer 3.3)}$$
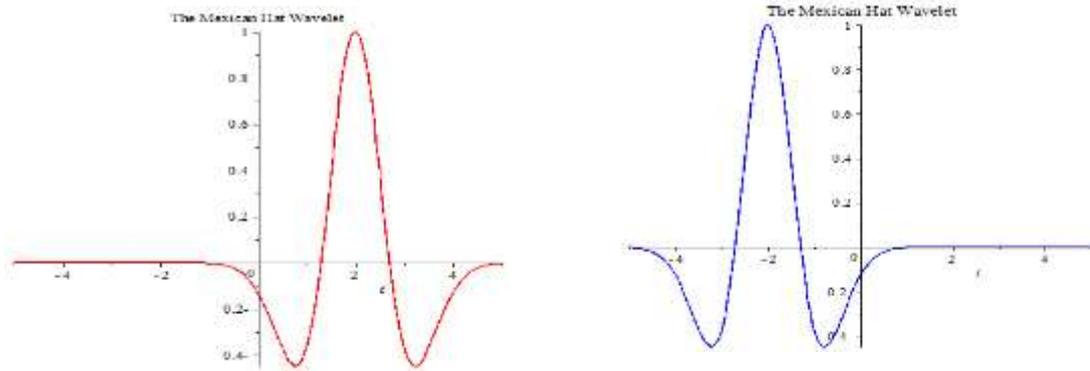
1) $\psi_{1,2=(1-2*(x-2)^2)e^{-(x-2)^2}}$

2) $\psi_{1,-2=(1-2*(x+2)^2)e^{-(x+2)^2}}$
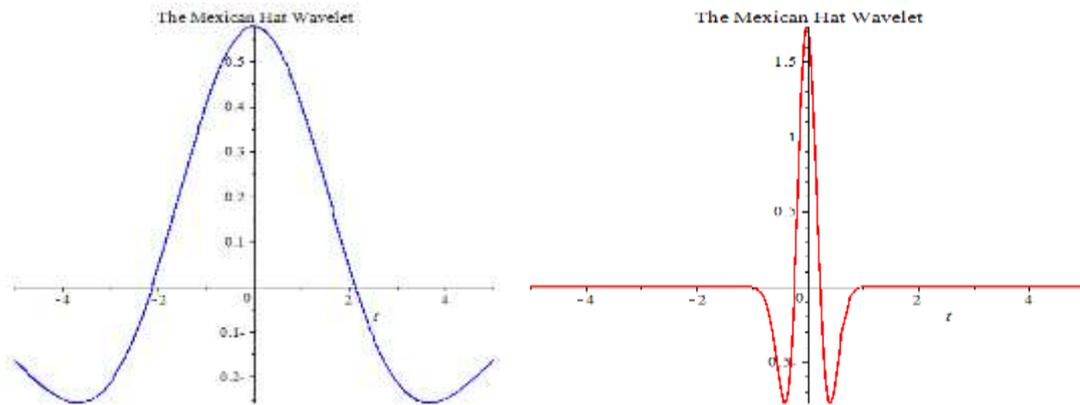
3) $\psi_{3,0=\frac{1}{\sqrt{3}}\left(1-2*\left(\frac{x}{3}\right)^2\right)e^{-\left(\frac{x}{3}\right)^2}}$

4) $\psi_{\frac{1}{3},0=\sqrt{3}(1-2*(3*x)^2)e^{-(3*x)^2}}$

**Figure 3.1:** graph of the Mexican hat wavelet

Figure 3.1 (1 , 2) is the graph of the Mexican hat wavelet where a =1 and b = 2 , a = 1 and b = -2, respectively, also fixing b = 0 and the scaling factor (or dilation) parameter a $\in \mathbb{R}$ and a $\neq 0$

$$\psi_{a,0} = \frac{1}{\sqrt{|a|}}\psi\left(\frac{x}{a}\right) = \frac{1}{\sqrt{|a|}}\psi(1-2\left(\frac{x}{a}\right)^2)e^{-\left(\frac{x}{a}\right)^2}$$

Notice that, the scaling factor a > 1 stretch's the wave see figure 3, while

0 < a < 1 shrinks the wave see figure 4

Definition 3.4 [4, 6]: Inverse wavelet transform

Let $\Psi(w)$ be the Fourier transform of the $\psi(x)$.

$$\Psi(w) = \int_{-\infty}^{\infty} \psi(x)e^{-iwx}dx \tag{3.9}$$

If $W_\psi f(a, b)$ is a CWT of a function $f(x)$ with a mother wavelet $\psi(x)$, then the inverse CWT is given by

$$f(x) = \frac{1}{c_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{|a|^2} W_\psi f(a, b)\psi_{a,b}(x)dadb. \tag{3.10}$$

where c_ψ is a constant that depends on the choice of the wavelet and is given by

$$c_\psi = \int_{-\infty}^{\infty} \frac{|\Psi(w)|^2}{|w|} dw < \infty \tag{3.11}$$

The inverse CWT exists if c_ψ is positive and finite.

Definition 3.5[3]: let ψ be a smooth function that is defined on $\mathbb{R}$ and has compact support. For $j, k \in \mathbb{Z}$ define a function $\psi_{j,k}$ by

$$\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k), \quad x \in \mathbb{R} \tag{3.12}$$

The function $\psi_{j,k}$ is a scaling by j units and a translation by k units of the mother wavelet ψ.

Example 3.2: To show the effect of scaling and translating, consider the function $\psi(x) = x^2$.

Use scaling and translations functions.

$$\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k) = 2^{\frac{j}{2}}(2^j x - k)^2 \ , x \in \mathbb{R} \ , j, k \in \mathbb{Z}$$

To understand the scaling role of j, fix $k = 0$ than

$$\psi_{j,0}(x) = 2^{\frac{j}{2}}\psi(2x) = 2^{\frac{j}{2}}(2^j x)^2 \ , j \in \mathbb{Z}, x \in \mathbb{R}$$
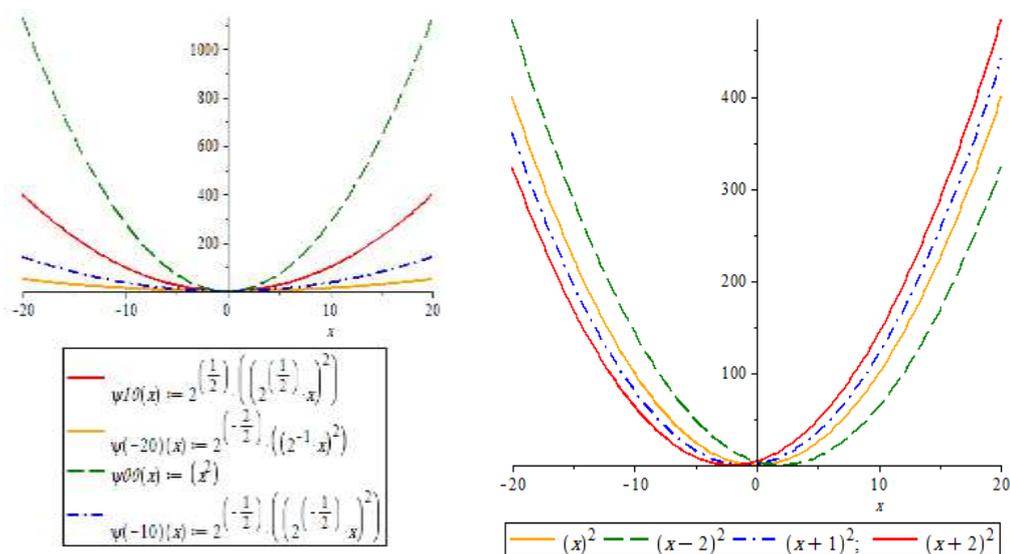
Figure 3.2: (a) (b)

(a) The scaling role of j, fix $k = 0$, in " mother wavelet "

(b) The translation role of k , fix $j = 0$, in " mother wavelet "

In figure 3.2(a), if j positive then the graph of $\psi_{j,0}$ is similar to the graph compressed, but if j is negative than the graph of $\psi_{j,0}$ is similar to the graph of ψ but has localized as is figure 3.2(a) .

also, to understand the translation role of k .

Fix $= 0$.

$$\psi_{0,k} = \psi(x - k) = (x - k)^2 \ , k \ \in \mathbb{Z} \ , x \in \mathbb{R} .$$

In figure 3.2(a), if k is positive then the graph to $\psi_{0,k}$ is similar to the graph of ψ but, shifted to the right, if k is negative, then the graph of $\psi_{0,k}$ is similar to the graph of ψ but shift to the left as in figure 3.2(b) .

Definition 3.6[3]: The wavelet system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ from orthonormal basis for $L^2(\mathbb{R})$ where

$$\psi_{j,k} = 2^{\frac{j}{2}}\psi(2^j x - k), \ \ x \in \mathbb{R} \tag{3.13}$$

$$w_{j,k} = \langle f(x), \psi_{j,k}(x) \rangle_{n\,n} = 2^{\frac{j}{2}} \ w_\psi f\left(\frac{1}{2^j}, 2^{\frac{k}{j}}\right) \tag{3.14}$$

Theorem 3.1[3] : The system $\{\psi_{j,k}(x)\}_{j,k\in\mathbb{Z}}$ is orthogonal on $(-\infty, \infty)$

$$\int_{-\infty}^{\infty} \psi_{j,k}(x)\psi_{\hat{j},\hat{k}}(x)dx = \begin{cases} 1 & if \ j = \hat{j}, k = \hat{k} \\ 0 & o.w \end{cases} \tag{3.15}$$

3.1.2 Discrete wavelet transforms (DWT)

Definition 3.7[9]: Discrete wavelet transform

In the CWT, replace the scaling parameter a by 2^{-j} and the translation parameter b by $k \ 2^{-j}$. Integral (3.2) becomes

$$w_\psi f(2^{-j}, k \ 2^{-j}) = 2^{-\frac{j}{2}} \int_{-\infty}^{\infty} f(x)\psi(2^j x - k)dx \tag{3.16}$$

3.1.3 Wavelet series

Under certain conditions, the series expansion of a function in terms of a given set of simple function $\{f_n\}_{n=0}^{\infty}$ is

$$f(x) = \sum_{n=0}^{\infty} a_n f_n(x) \tag{3.17}$$

The function and the set $\{f_n\}_{n=0}^{\infty}$ must satisfy certain conditions that enable us finding the coefficients a_n. For example Taylor series expands a periodic function of period $p = L$ that is analytic at point x_0 using the orthogonal system $\{(x - x_0)^n\}_{n=0}^{\infty}$ while Fourier series expands a square integrable function using the orthogonal trigonometric system $\left\{ sin\left(\frac{n\pi x}{L}\right), cos(\frac{n\pi x}{L}) \right\}_{n=0}^{\infty}$. *It is* the orthogonality of such system on $[-L, L]$ that enable us to compute the coefficients in the expansion of the function.

Definition 3.8[3]: Wavelet series

The wavelet series expansion of a given function f is given by:

$$f(x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{j,k} \psi_{j,k}(x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k} \rangle f_{j,k} \tag{3.18}$$

where the coefficients $c_{j,k}$ or $\langle f, \psi_{j,k} \rangle$ are uniquely defined by

$$c_{j,k} = \langle f, \psi_{j,k} \rangle = \int_{-\infty}^{\infty} f(x) \overline{\psi_{j,k}(x)} dx \tag{3.19}$$

where $\psi_{j,k} = 2^j \psi(2^j x - k)$.

The wavelet system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is fully determined by the function ψ, that is, we only need to store, information about the single function ψ to be

used in (3.19). For existence of wavelet series expansion, the mother wavelet must satisfy the admissibility conditions given in definition 3.1

In practice, we don't expect ψ to have all the above properties, so we need to be careful and only insist on the properties that are needed in the application we have in mind.

In cases where we need to work with the time behavior of functions as well as their frequency-content, we must replace the wish for compact support of either ψ or $\overline{\psi}$ with the requirement that the function at last tends to zero very fast. Formulated for the function ψ, such a requirement could be that there exist constants $c, \alpha > 0, s.t$

$$|\psi(x)| \leq ce^{-\alpha|x|} \ , \forall x \in \mathbb{R}$$

Assuming the convergence of the wavelet series, we may use it to approximate a given function by :

$$f(x) \approx \sum_{j=-n} \sum_{k=-N} C_{j,k} \psi_{j,k}(x)$$

For sufficiently large value of $n \in N$ [3]

3.2 Advantages of Wavelet Theory [8]

1. The most advantage of wavelets is that they show a simultaneous localization in time and frequency domain.

2. The next advantage of wavelets is the speed fast wavelet transform.

3. Wavelets have the great advantage of being able to separate the fine details in a signal. Very small wavelets can be used to isolate very fine details in a signal, while very large wavelets can identify coarse details.

4. A wavelet transform can be used to decompose a signal into component wavelets.

5. In wavelet theory, it is often possible to obtain a good approximation of the given function *f* by using only a few coefficients which is the great achievement in comparison to Fourier transform.

6. Wavelet theory is capable of revealing aspects of data like trends, breakdown points, and discontinuities in higher derivatives and self-similarity.

7. It can often compress or de-noise a signal without appreciable degradation.

3.3 Comparison of wavelet transform with Fourier transform

1. Fourier transform is a strong tool for analyzing a collection of stationary signals (a signal with no change in the properties). Sine and cosine (sinusoid signals), for example, are processed by the application of Fourier transform. However, the analysis of non-stationary signals (where the change in the properties occurs) is less useful by Fourier application. On the contrary, wavelet transform is applicable to both stationary and non-stationary signals [7, 8].

2. Fourier transform proves to be useful in certain areas out of traditional signal processing. But it should be noted that the mathematical design of wavelets is broader than the Fourier transform, and to be more specific, the mathematics of wavelets include Fourier transform [7, 8].

3. Wavelet and Fourier transforms are mainly different in the domains of time and frequency. Wavelets are well localized in time and frequency, while Fourier application in such domains is standardized in localizing them. While both have good effects out of localized time and frequency, wavelet shows better representation by Walnut multi-resolution analysis [7, 8, 9].

4. Fourier transform relies on a single scaled function $\psi(x)$, but the wavelet transform has the ability to move the function and generate a two-parameter series of functions $\psi_{a,b}(x)$ defined [8, 9].

Chapter Four

Haar wavelet

Wavelet transform or wavelet analysis is recently developed mathematical applications for many problems. There are many wavelets. One of the most simple and popular is the Haar wavelets.

Because of simplicity of Haar wavelet, it becomes an efficient technique for solving many problems that arises in many branches of science, engineering and economic. So, in the next chapter, we will use wavelets for solving differential equations. Haar function was first introduced by the Hungarian mathematician Alfred Haar in appendix of his doctoral thesis in 1910, and later, it was developed by others. Haar function is an odd rectangular pulse pair, is the simplest and oldest orthonormal wavelet with compact support. There are different definitions of Haar function and various generalizations have been used [9, 25].

4.1 Haar Wavelet function

4.1.1 The Haar scaling function

There are two functions that play a primary role in wavelet analysis, the scaling function ϕ and the wavelet ψ. These two functions generate a family of functions that can be used to break up or reconstruct signals. To emphasize the family involving ϕ and ψ, ϕ is called the "father wavelet " and ψ is the "mother wavelet".

Example 4.1: The building blocks are translations and dilations (both in height and width) in basic graph [2].
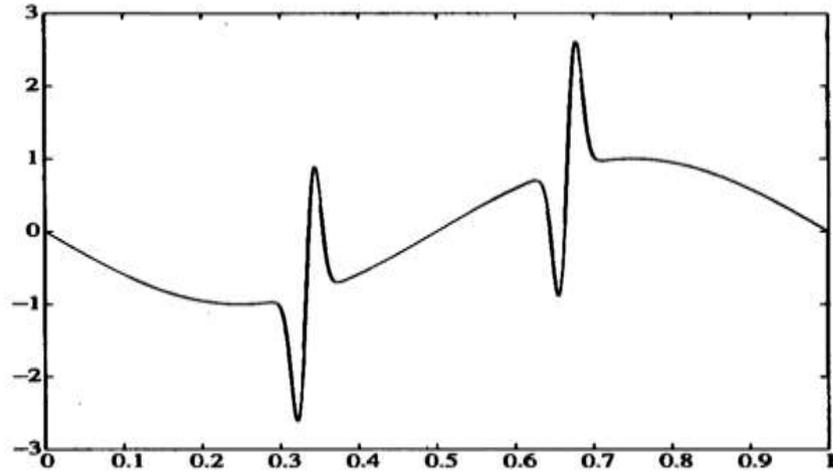
**Figure 4.1:** Voltage from a faulty meter
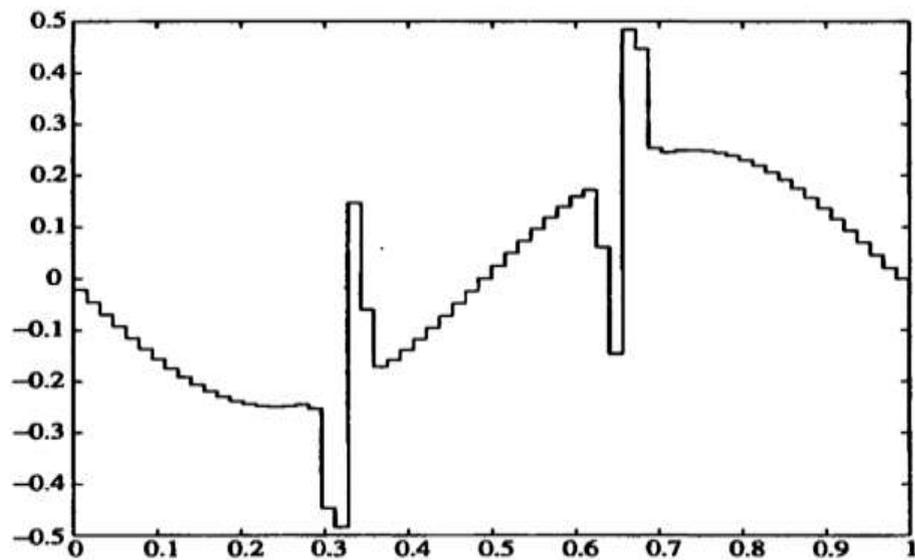
Figure 4.2: Approximation voltage signal by Haar function

The building blocks generated by the Haar scaling function are particularly simple and illustrate the general idea underlying a multire- solution analysis, which we will discuss next. The disadvantage of the Haar

wavelets is that, they are discontinuous and therefore do not approximate continuous signals very well.

4.1.2 Basic properties of the Haar scaling function

Definition 4.1[2]: The Haar scaling function, also called the father wavelet, is defined as

$$\phi(x) = \begin{cases} 1 & if \ 0 \le x < 1 \\ 0 & else \ where \end{cases} \tag{4.1}$$
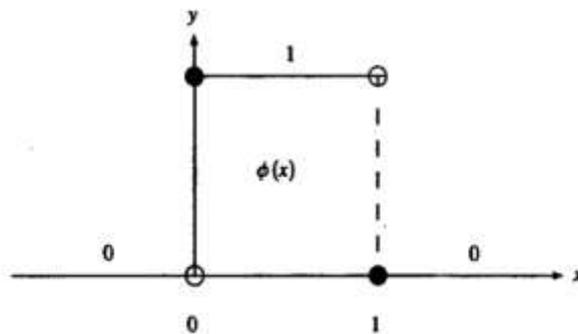
Figure 4.3: Graph of the Haar scaling function.

Remark [2, 5]: The function $\phi(x - k)$ has the same graph as ϕ but translated to the right by k units (assuming k is positive).

Notation: we will use V_0 for the space of all functions of the form

$$\sum_{k \in \mathbb{Z}} a_k \phi(x - k) \qquad \qquad a_k \in \mathbb{R} \tag{4.2}$$

where k ranges over any finite set of positive or negative integers.

Remark [2]: since $\phi(x - k)$ is discontinuous at $x = k$ and $x = k + 1$, an alternative description of V_0 is that it consists of all piecewise constant functions with possible discontinuities at integers. Each element of V_0 has

finite or compact support which means the element is zero outside a bounded set. The graph of a typical element of V_0 is given in Figure 4.4
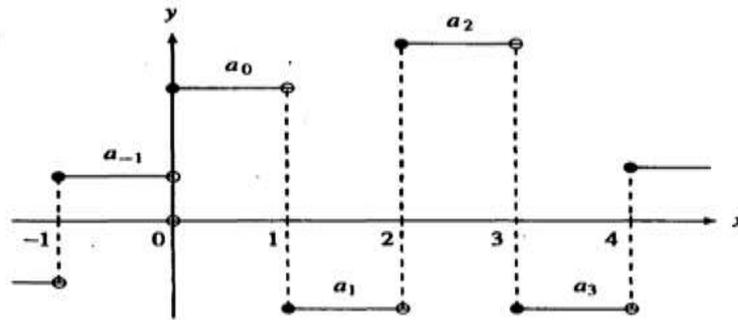
Figure 4.4: Graph of typical element of V_0.

Note that a function in V_0 may not have discontinuities at all integers for example, if $a_1 = a_2$ then the preceding sum is continuous at $x = 2$.

Example 4.2: The function

$$f(x) = 3\phi(x) - \phi(x - 1) + 2\phi(x - 2) + 2\phi(x - 3) - 2\phi(x - 4) \in V_0$$

has discontinuities at $x = 0,1,2,4$ and 5 but not at $x = 3$ see figure (4.5).
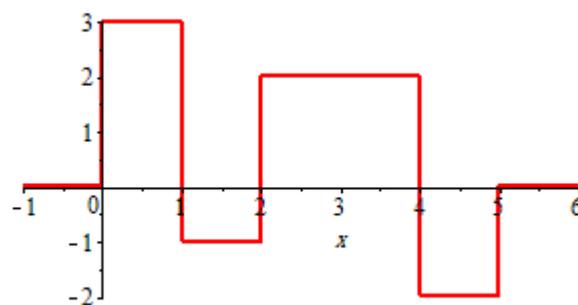
Figure 4.5: Plot of f in example 4.2

We need thinner blocks to analyze signals of high frequency. For example, the width of the building block $\phi(2x)$ is half the width of $\phi(x)$, see figure 4.6

Figure 4.6:

The function $\phi(2x - k) = \phi\left(2(x - \frac{k}{2})\right)$ is the same as the graph of the function of $\phi(2x)$ but shifted to right by $\frac{k}{2}$ units [2,5].

Notation: let V_1 be the space of functions of the form

$$\sum_{k\in\mathbb{Z}} a_k\phi(2x - k) \qquad\qquad a_k \in \mathbb{R} \qquad\qquad\qquad (4.3)$$

Geometrically, V_1 is the space of piecewise constant functions of finite support with possible discontinuities at half-integers $\{0, \pm\,^1/_2,\ \pm 1, \pm\,^3/_2,\ ...\}$.

Example 4.3: The function

$$f(x) = 3\phi(2x) - \phi(2x - 1) - \phi(2x - 2) + 2\phi(2x - 3) \ \in V_1$$

has discontinuities at $x = 0\ ,\frac{1}{2}\ ,1,2,\frac{5}{2}$ and 3 as shown in figure (4.7).
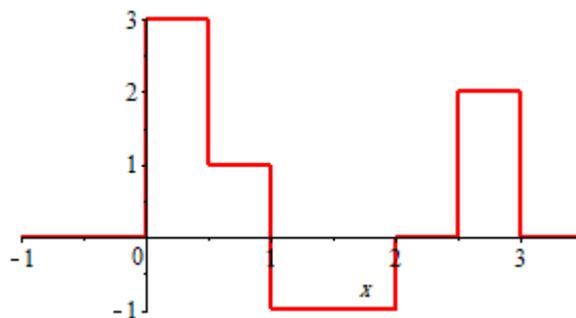
Figure 4.7:

Definition 4.2[2]: The space of V_0 is the space of all piecewise constant function with discontinuities contained in the set of integers. The space V_1 is the space of all piecewise consists functions with discontinuities contained in the set of half integers $\left\{\frac{2n+1}{2} : n \in \mathbb{Z}\right\}$.

Definition 4.3[2]: suppose j is any nonnegative integer. The space of step functions at level j, denoted by V_j is defined to be the space spanned by the set $\left\{..., \phi(2^j x + 1), \phi(2^j x), \phi(2^j x - 1), \phi(2^j x - 2), ...\right\}$ over The real numbers. V_j is the space of piecewise constant functions of finite support whose discontinuities are contained in the set .

$$\left\{..., {-1}/_{2^j}, 0, {1}/_{2^j}, {2}/_{2^j}, {3}/_{2^j}, ...\right\}.$$

The same applies for $V_1 \subset V_2$ and so forth :

$$V_0 \subset V_1 \subset \cdots V_{j-1} \subset V_j \subset V_{j+1} \cdots$$

Remark: This containment is strict. For example, the function $\phi(2x)$ belongs to V_1 but does not belong to V_0 [since $\phi(2x)$ is discontinuous at $x = \frac{1}{2}$].

Theorem 4.1[2]:

A function $f(x)$ belongs to V_0 if and only if $f(2^j x)$ belong to V_j.

A function $f(x)$ belongs to V_j if and only if $f(2^{-j} x)$ belongs to V_0.

Theorem 4.2[10]: A set of nonzero functions $\{\phi(x - k), k \in \mathbb{Z}\}$ in L^2 space are orthonormal to each V_j component.

$$\|\phi(x-k)\|_{L^2}^2 = \int_{-\infty}^{\infty} \phi(x-k)^2\, dx = \int_{k}^{k+1} 1\, dx = 1 \tag{4.4}$$

$$\langle \phi(x-j), \phi(x-k)\rangle_{L^2} = \int_{-\infty}^{\infty} \phi(x-j)\phi(x-k)dx = 0, \qquad j \neq k \tag{4.5}$$
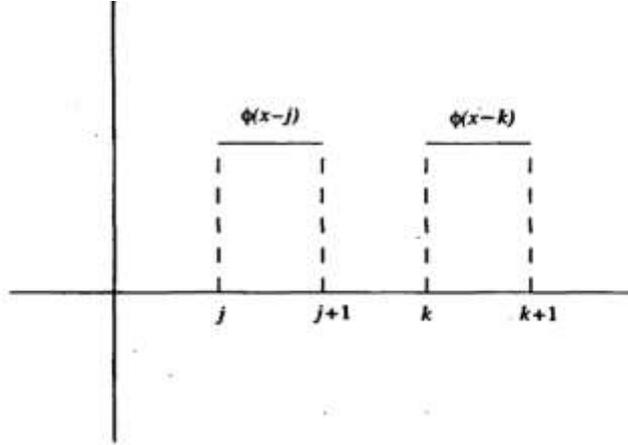
Figure 4.8: $\phi(x-j)$ and $\phi(x-k)$ have disjoint support.

Theorem 4.3[11]: The set of functions $\left\{ 2^{\frac{j}{2}} \phi(2^j x - k),\ k \in \mathbb{Z} \right\}$ is an orthonormal basis of V_j.

Definition 4.4[15]: The function

$$\psi(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1 & \frac{1}{2} \le x < 1 \\ 0 & otherwise \end{cases} \tag{4.6}$$

Is called the "mother wavelet ".

The Haar wavelet has compact support, and clearly $\int_{-\infty}^{\infty} \psi(x)dx = 0$ and

$\psi \in L^2(\mathbb{R})$, but this wavelet is not continuous.
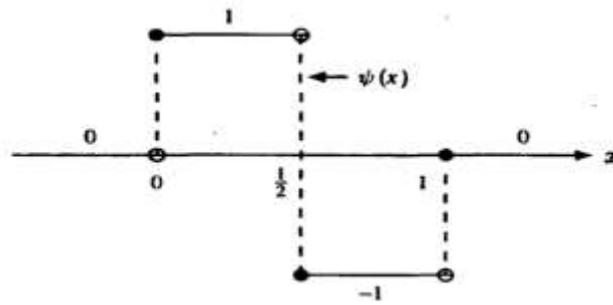
Figure 4.9: The Haar wavelet $\psi(x)$.

Figure 4.9 indicates that the Haar wavelet has good time localization but poor frequency localization. Most of applications of wavelet exploit their ability to approximate functions as efficiently as possible, which means as few coefficients as possible.

Definition 4.5 [9]: Vanishing Moment

The wavelet is said to have M ($M \in \mathbb{N}$) vanishing moment if it satisfies The following condition.

$$\int_{-\infty}^{\infty} x^m \psi(x) dx = 0, \quad m = 0,1,\ldots M - 1. \tag{4.7}$$

Remark [28, 30]: The smoothness of the wavelet increases as the number of vanishing moment increases.

Remark [28]: The Haar wavelet ψ has compact support on I if it vanishes outside I. If ψ has M vanishing moment, then its support is at least of length $2M - 1$, so the Haar wavelet has minimum support equal to one.
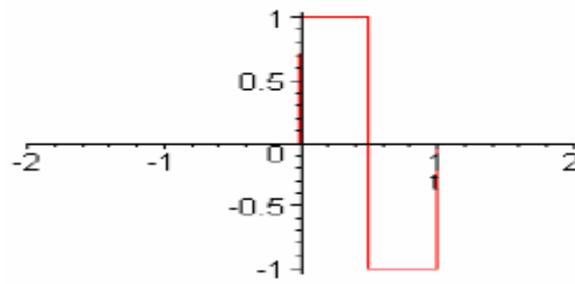
Figure 4.10: plot of the mother wavelet.

Definition 4.6[14]: a function in V_1 is orthogonal to V_0 if and only if it is of the form $\sum_{k\in\mathbb{Z}} a_k\,\psi(x-k), \quad a_k \in \mathbb{R},$ (4.8)

let W_0 be the space of all functions of the form

$$\sum_{k\in\mathbb{Z}} a_k\psi(x-k), \qquad a_k \in \mathbb{R},$$

where, again, we assume that only a finite number of a_k are nonzero. Then W_0 is the orthogonal complement of V_0 in V_1; in other words,

$$V_1 = V_0\oplus W_0$$

Theorem 4.4[2]: let W_j be the space of functions of the form

$$\sum_{k\in\mathbb{Z}} a_k\psi(2^j x - k) \qquad a_k \in \mathbb{R}$$ (4.9)

where we assume that only a finite number of the coefficients a_k are nonzero. W_j is the orthogonal complement of V_j in V_{j+1} and

$$V_{j+1} = V_j\oplus W_j$$ (4.10)

Definition 4.7[2]: The Haar wavelet can be written as

$$\psi(x) = \phi(2x) - \phi(2x - 1)$$ (4.11)

4.2 Properties of Haar wavelet.

1. Haar wavelet is very well localized in the x domain, but not continuous.

2. $\int_0^\infty \psi(x)\, dx = 0$ and $\int_0^\infty |\psi(x)|^2\, dx = 1$ $\hspace{2cm}$ (4.12)

3. Any continuous real function can be approximated by linear combination of $\phi(x)$, $\phi(2x)$, $\phi(4x), \ldots, \phi(2^j x)$ and their shifted functions.

This extends the function space where any function can be approximated by continuous functions.

4. Any continuous real function can be approximated by linear combination of the constant function $\psi(x)$, $\psi(2x)$, $\psi(4x), \ldots, \psi(2^j x)$ and their shifted functions.

5. A Wavelet function can be written as a combination of wavelet functions with different scales: $\psi(x) = \phi(2x) - \phi(2x - 1)$ $\hspace{2cm}$ (4.13)

Similarly, for the scaling function: $\phi(x) = \phi(2x) + \phi(2x - 1)$ (4.14)

6. Haar function are orthogonal.

i.e. $\int_{-\infty}^\infty 2^{j.m}\, \psi(2^j x - k)\psi(2^m x - n)dx = \delta_{j.m}\, \delta_{k.n}$ $\hspace{1.5cm}$ (4.15)

7. The integration of Haar wavelets can be expandable into Haar series.

4.3 Wavelet collocation method

Collocation method [9] used in numerical solution of differential equations. The main idea involves numerical operators acting on point values (collocation points) in the physical space, or dividing the domain as a

number of points and calculates the solution about it. Generally, wavelet collocation methods are created by choosing a wavelet and some kinds of grid structure which will be computationally adapted.

In effect, one obtains finites differences on non-uniform grid. The treatment of nonlinearities in wavelet collocation method is straight forward task.

4.4 Haar wavelet transformation

The Haar wavelet is system [15, 17].

$$h_i(x) = 2^{\frac{j}{2}}\psi(2^j x - k) = \begin{cases} 1 & for \ x \in [\varepsilon_1, \varepsilon_2) \\ -1 & for \ x \in [\varepsilon_2, \varepsilon_3) \\ 0 & otherwise \end{cases} \tag{4.16}$$

where $\varepsilon_1 = \dfrac{k}{m}$, $\qquad \varepsilon_2 = \dfrac{k+0.5}{m}$, $\qquad \varepsilon_3 = \dfrac{k+1}{m}$.

where $m = 2^j$, $j = 0,1,2, ...J$, $k = 0,1,2, ... , m-1$

j indicates the level of wavelet or the dilation parameter or scaling (changing levels).

k denotes translation parameter.

J denotes the maximum level of resolution.

The index i in $h_i(x)$ is determined by $i = m + k + 1$. In the case of minimal values $m = 1$, $k = 0$, we have $i = 2$. The maximum value of i is $i = 2M = 2^{J+1}$.

For $i = 1$, the function $h_1(x)$ is the scaling function or the father wavelet for the family of the Haar wavelets which is defined as

$$h_1(x) = \begin{cases} 1 & x \in [0,1) \\ 0 & otherwise \end{cases} \qquad (4.17)$$

For $i = 2$, the function $h_2(x)$ is the mother wavelet for the family of the Haar wavelet which is defined as

$$h_2(x) = \begin{cases} 1 & 0 \leq x < 1/2 \\ -1 & 1/2 \leq x < 1 \\ 0 & otherwise \end{cases} \qquad (4.18)$$

For $i = 3$, the function $h_3(x)$ is defined as

$$h_3(x) = \begin{cases} 1 & x \in \left[0, \frac{1}{4}\right) \\ -1 & x \in \left[\frac{1}{4}, \frac{2}{4}\right) \\ 0 & otherwise \end{cases} \qquad (4.19)$$

For $i = 4$, the function $h_4(x)$ is defined as

$$h_4(x) = \begin{cases} 1 & x \in \left[\frac{2}{4}, \frac{3}{4}\right) \\ -1 & x \in \left[\frac{3}{4}, \frac{4}{4}\right) \\ 0 & otherwise \end{cases} \qquad (4.20)$$

Theorem 4.5[16]: Haar wavelets are orthogonal to each other and therefore constitute an orthogonal basis shown as.

$$\int_0^1 h_i(x)h_l(x)dx = \begin{cases} 2^{-j} & i = l = 2^j + k + 1 \\ 0 & i \neq l \end{cases} \qquad (4.21)$$

Table 4.1: Index computations for Haar basis function

j	0	1	1	2	2	2	2	3	3	3	3	3	......
k	0	0	1	0	1	2	3	0	1	2	3	4	.....
$i = 2^j + k + 1$	2	3	4	5	6	7	8	9	10	11	12	13	.....
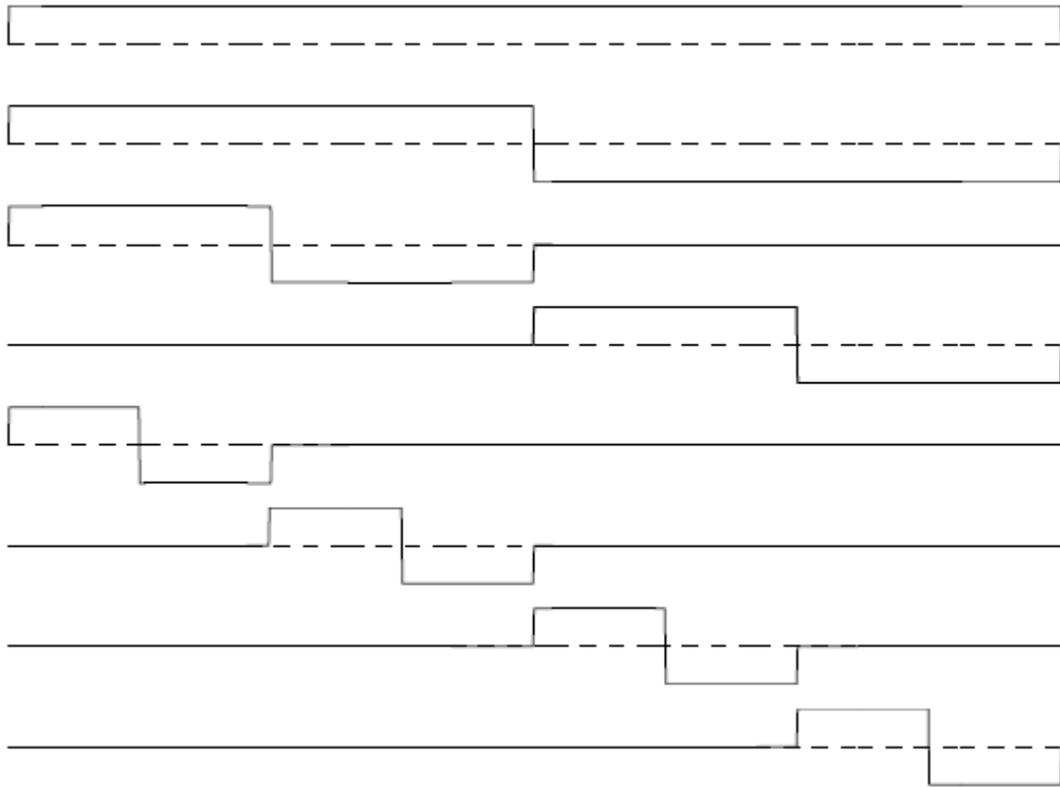
Figure 4.11: The first eight Haar functions

The orthogonal set of Haar wavelets from $h_1(x)$ to $h_8(x)$ is shown in figure 4.11, which contains a family of single square wavelets. The first basis $h_1(x)$ is called scaling function, which is equal to one for whole unit time interval. The second basis $h_2(x)$ is the fundamental square wave is called mother wavelet. The others, $h_3(x)$ to $h_8(x)$ are generated from $h_2(x)$ via two operations: dilation and translation.

4.5 Function approximation

Any function $u(x) \in L^2[1,0)$ can be decomposed as [17]

$$u(x) = \sum_{i=0}^{\infty} a_n h_n(x), \quad i \in \{0\} \cup \mathbb{N}, \tag{4.22}$$

where the coefficients a_n are determined as,

$$a_0 = \int_0^1 u(x)h_0(x)dx, \quad a_n = 2^j \int_0^{44} u(x)h_i(x)dx \qquad (4.23)$$

where $n = 2^j + k, \quad j \geq 0, \quad 0 \leq k \leq 2^j, \quad x \in [0,1)$

such that the following error norm ϵ :

$$\epsilon = \int_0^1 \left[u(x) - \sum_{i=0}^{m-1} a_i h_i(x) \right]^2 dx, \qquad m = 2^j, \quad j \in \{0\} \cup \mathbb{N} \ (4.24)$$

is minimized

Usually, the series expansion of $u(x)$ contains infinite number of terms. if $u(x)$ is piecewise constant by itself, or many be approximated as piecewise constant during each subinterval.

4.6 Convergence analysis of Haar wavelets

Assume that $f(x)$ is a differentiable function with

$$|f(x)| \leq K, \qquad \forall x \in (a,b) \qquad such\ that \qquad |f'(x)| \leq K.$$

where K is a positive constant. Haar wavelet approximation for the function $f(x)$ is given by

$$f_M(x) = \sum_{i=1}^{2M} a_i h_i(x) \qquad (4.25)$$

Babolian and Shahsavaran [21] have shown that the square of the error norm for wavelet approximation is given by

$$\|f(x) - f_M(x)\|^2 = \frac{K^3}{12M^2} \qquad (4.26)$$

Therefore, $\|f(x) - f_M(x)\| = O\left(\frac{1}{M}\right).$

The maximum absolute error $= L_\infty = \max.|u_l^e(x) - u_l^a(x)|$ (4.27)

The maximum relative error $= L_{mre} = \dfrac{L_\infty}{|u_l^e(x)|}$ (4.28)

where u_l^e and u_l^a are the exact and Haar solution respectively at the lth collocation points x_l, $l = 1, 2, \dots, 2m$. We choose the computational domain $[\,0\,,1\,]$ for each numerical example.

4.7 Integration of Haar wavelets

We want to integrate differential equations following the method of Chen and Hsiao method (CHM) and we'll talk about this principal method more clearly in the next chapter. We have to evaluate the integrals[16].

$$P_{i,\alpha}(x) = \underbrace{\int_A^x \int_A^x \;\cdots\; \int_A^x h_i(\tau)\,d\tau^\alpha}_{\substack{(\alpha-time) \\ \alpha=1,2,\dots,n,\;\; i=1,2,\dots\;2M}} = \frac{1}{(\alpha-1)} \int_A^x (x-\tau)^{\alpha-1} h_i(\tau)\,d\tau \;(4.29)$$

The case $\alpha = 0$ corresponds to the function $h_i(\tau)$.

Taking account of $h_i(x)$, these integrals can be calculated analytically. By doing so, we obtain

$$P_{\alpha,i}(x) = \begin{cases} 0 & for\ x < \varepsilon_1 \\ \frac{1}{\alpha!}[x - \varepsilon_1]^\alpha & for\ x \in [\varepsilon_1, \varepsilon_2] \\ \frac{1}{\alpha!}\{[x - \varepsilon_1]^\alpha - 2[x - \varepsilon_2]^\alpha\} & for\ x \in [\varepsilon_2, \varepsilon_3] \\ \frac{1}{\alpha!}\{[x - \varepsilon_1]^\alpha - 2[x - \varepsilon_2]^\alpha + [x - \varepsilon_3]\} & for\ x > \varepsilon_3 \end{cases} \quad (4.30)$$

These formulas hold for $i > 1$. In the case $i = 1$, we have $\varepsilon_1 = A$, $\varepsilon_2 = \varepsilon_3 = B$, and

$$P_{1.\alpha}(x) = \frac{1}{\alpha!}(a - A)^{\alpha}.$$

In the Haar wavelet method [16], the following integrals are used

$$P_{i,1}(x) = \int_0^x h_i(\tau)d\tau \qquad\qquad (4.31)$$

$$P_{i,1}(x) = \begin{cases} x - \varepsilon_1 & x \in [\varepsilon_1, \varepsilon_2) \\ \varepsilon_3 - x & x \in [\varepsilon_2, \varepsilon_3) \\ 0 & otherwise \end{cases} \qquad\qquad (4.32)$$

$$P_{i,v+1}(x) = \int_0^x P_{i,v}(\tau)d\tau \quad , \qquad v = 1,2,3, \dots. \qquad\qquad (4.33)$$

Some calculated $P_{i,v+1}(x)$, $v = 1,2$, ... are given below

$$P_{i,2}(x) = \begin{cases} 0 & x \in [0, \varepsilon_1) \\ \frac{(x-\varepsilon_1)^2}{2} & x \in [\varepsilon_1, \varepsilon_2) \\ \frac{1}{4m^2} - \frac{(\varepsilon_3-x)^2}{2} & x \in [\varepsilon_2, \varepsilon_3) \\ \frac{1}{4m^2} & x \in [\varepsilon_3, 1) \end{cases} \qquad\qquad (4.34)$$

$$P_{i,3}(x) = \begin{cases} 0 & x \in [0, \varepsilon_1) \\ \frac{(x - \varepsilon_1)^3}{6} & x \in [\varepsilon_1, \varepsilon_2) \\ \frac{(x - \varepsilon_2)}{4m^2} - \frac{(\varepsilon_3 - x)^3}{6} & x \in [\varepsilon_2, \varepsilon_3) \\ \frac{x - \varepsilon_2}{4m^2} & x \in [\varepsilon_3, 1] \end{cases} \qquad\qquad (4.35)$$

$$P_{i,4}(x) = \begin{cases} 0 & x \in [0, \varepsilon_1) \\ \frac{(x - \varepsilon_1)^4}{24} & x \in [\varepsilon_1, \varepsilon_2) \\ \frac{(x - \varepsilon_2)}{8m^2} - \frac{(\varepsilon_3 - x)^4}{24} + \frac{1}{192m^4} & x \in [\varepsilon_2, \varepsilon_3) \\ \frac{x - \varepsilon_2}{8m^2} + \frac{1}{192m^4} & x \in [\varepsilon_3, 1] \end{cases} \qquad\qquad (4.36)$$

Similarly for $P_{i,v}(x)$, $v = 5,6, ...,$ are obtained in similar way.

4.8 The product operational matrix of the Haar wavelet

The integrals in (4.32- 4.36) are evaluated can be used for all differential equations [14].

It is convenient to pass to the matrix formulation. For this purpose the interval $x \in [0, 1]$ is divided into $2m$ parts each of length $\Delta x = \frac{1}{2m}$, where $m = 2^j$. We define the collocation points where we will approximate any function as

$$x_l = \frac{2l - 1}{4m}, \qquad for \ l = 1, 2, ..., 2m \tag{4.37}$$

At these collocation points, we discretize the Haar function $h_i(x)$ to obtain $H_{2m \times 2m}$ coefficient matrix called Haar matrix.

The Haar coefficient matrix H is defined as

$$H_{(m)}(i, l) = h_i(x_l) \tag{4.38}$$

To compute H, we define a vector of Haar functions

$$h_{(m)}(x) = [h_0(x), h_1(x), ... h_{m-1}(x)]^T \tag{4.39}$$

where m is the dimension of the vector. These vectors are used to compute each column of Haar matrix as

$$H_{(m)} = \left[h_{(m)}\left(\frac{1}{4m}\right), h_{(m)}\left(\frac{3}{4m}\right), ..., h_{(m)}\left(\frac{(4m - 1)}{4m}\right) \right]^T \tag{4.40}$$

The integral matrices P_v have the elements $P_v(i, l) = P_{i,v}(l)$.

Chen and Hsiao defined the integral matrix in a different way. They calculated $P_{(m)}$ from the equation

$$\int_0^x h_{(m)}(\tau)d\tau = P_{(m)}h_{(m)}(x), \quad x \in [0,1) \tag{4.41}$$

The result is the square matrix $P_{(m)} = P_{(m \times m)}$ which is called the operational matrix of integration [13].

Chen and Hsiao method showed that the following recursive formula holds:

$$P_{(m)} = \frac{1}{2m}\begin{bmatrix} 2m\,P_{\left(\frac{m}{2}\times\frac{m}{2}\right)} & -H_{\left(\frac{m}{2}\times\frac{m}{2}\right)} \\ H^{-1}_{\left(\frac{m}{2}\times\frac{m}{2}\right)} & O_{\left(\frac{m}{2}\times\frac{m}{2}\right)} \end{bmatrix} \tag{4.42}$$

where $O_{\frac{m}{2}\times\frac{m}{2}}$ is null matrix.

$$H_{(m)} = \left[h_{(m)}(x_0), h_{(m)}(x_1), \dots, h_{(m)}(x_{m-1})\right]^T \tag{4.43}$$

where $\frac{l}{m} \leq x_l \leq \frac{l+1}{m}$ and $H^{-1}_{(m \times m)} = \frac{1}{m}H^T_{(m \times m)}diag(r)$

It should be noted that calculations for $P_{(m)}$ and $H_{(m)}$ must be carried out only once. Since H and H^{-1} contain many zeros, this phenomenon makes the Haar transform faster than Fourier transform, this is one of the reasons for rapid convergence of Haar wavelet series [19].

Then $u(x)$ will terminate at finite terms [17].

$$u(x) \cong \sum_{i=0}^{m-1} a_n\, h_n(x) = a^T_{(m)}h_{(m)}(x) \tag{4.44}$$

where the coefficient vector $\boldsymbol{a}_{(m)}^T$ and the Haar function vector $\boldsymbol{h}_{(m)}(x)$ are defined as

$$\boldsymbol{a}_{(m)}^T = [a_0, a_1, \dots a_{m-1}] \quad \text{and} \quad \boldsymbol{h}_{(m)}(x) = [h_0(x), h_1(x), \dots h_{m-1}(x)]^T$$

Example 4.4: Haar matrices

We start with Haar matrix of order one: $H_{(1)} = [1]$

1. Haar matrix of order two

The first two Haar function vectors with $x = \frac{1}{4}, l = 1,3$ can be expressed the following:

$$h_{(2)}\left(\frac{1}{4}\right) = \left[h_0\left(\frac{1}{4}\right), h_1\left(\frac{1}{4}\right)\right]^T = [1,1]^T$$

$$h_{(2)}\left(\frac{3}{4}\right) = \left[h_0\left(\frac{3}{4}\right), h_1\left(\frac{3}{4}\right)\right]^T = [1,-1]^T$$

which can be written in matrix from as

$$H_{(2)} = \left[h_{(2)}\left(\frac{1}{4}\right), h_{(2)}\left(\frac{3}{4}\right)\right]^T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

2. Haar matrix of order four

The first four Haar function vectors with $x = \frac{1}{8}, l = 1,3,5,7$ can be expressed as follows:

$$h_{(4)}\left(\frac{1}{8}\right) = [h_0\left(\frac{1}{8}\right), h_1\left(\frac{1}{8}\right), h_2\left(\frac{1}{8}\right), h_3\left(\frac{1}{8}\right)]^T = [1,1,1,0]^T$$

$$h_{(4)}\left(\frac{3}{8}\right) = \left[h_0\left(\frac{3}{8}\right), h_1\left(\frac{3}{8}\right), h_2\left(\frac{3}{8}\right), h_3\left(\frac{3}{8}\right)\right]^T = [1,1,-1,0]^T$$

$$h_{(4)}\left(\frac{5}{8}\right) = [h_0\left(\frac{5}{8}\right), h_1\left(\frac{5}{8}\right), h_2\left(\frac{5}{8}\right), h_3\left(\frac{5}{8}\right)]^T = [1,-1,0,1]^T$$

$$h_{(4)}\left(\frac{7}{8}\right) = [h_0\left(\frac{7}{8}\right), h_1\left(\frac{7}{8}\right), h_2\left(\frac{7}{8}\right), h_3\left(\frac{7}{8}\right)]^T = [1, -1, 0, -1]^T$$

which can be written in matrix from as

$$H_{(4)} = \left[h_{(4)}\left(\frac{1}{8}\right), h_{(4)}\left(\frac{3}{8}\right), h_{(4)}\left(\frac{5}{8}\right), h_{(4)}\left(\frac{7}{8}\right)\right]^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

3. The same method can be written as Haar matrix of order eight.

$$H_{(8)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Example 4.5: The operational matrix of integration shown as

We start with the operational matrix of order one: $P_{(1)} = \begin{bmatrix} \frac{1}{2} \end{bmatrix}$

1. The operational matrix of order two is obtained by the recursive formula and the operational matrix of rank one defined as

$$P_{(2)} = \frac{1}{4}\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

2. The operational matrix of order four is obtained by the recursive formula and the operational matrix of rank two defined as

$$P_{(4)} = \frac{1}{16} \begin{bmatrix} 8 & -4 & -2 & -2 \\ 4 & 0 & -2 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

3. The operational matrix of order eight is obtained by the recursive formula and the operational matrix of rank four defined as

$$P_{(8)} = \frac{1}{64} \begin{bmatrix} 32 & -16 & -8 & -8 & -4 & -4 & -4 & -4 \\ 16 & 0 & -8 & 8 & -4 & -4 & 4 & 4 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Chapter Five

Wavelet method for differential equations

Haar wavelets functions appear very attractive in many applications as in solving differential equations, image processing, data compression and computer graphics. In this thesis, we use Haar wavelet method for solving differential equations. In this chapter, we consider second-order linear ordinary differential equations and linear partial differential equations.

5.1 The method of solution for differential equations

The Haar wavelet is the simplest orthonormal wavelet with compact support. It should be mentioned that the Haar wavelet has a fundamental imperfection, it is not continuous. At points of discontinuity the derivatives do not exist, therefore it is not possible to apply the Haar wavelet directly for solving differential equations.

There are two ways to avoid the discontinuity problem. First, The piecewise constant Haar functions can be regularized with interpolation splines. This technique has been applied by Cattani. This greatly complicates the solutions process, losing the main advantage of Haar wavelets, namely its simplicity [23]. The second possibility, the method we adopt, is the use of Chen and Hsiao method. They recommend expanding the highest derivative appearing in the differential equation in terms of its Haar series instead of expanding the solution. Then the expansion is integrated a number of times equals the order of the highest derivative to obtain the expansion of the solution and all of its derivatives through

integration. In the power series method, we start with the power series expansion of the solution, and then we obtain the expansion of the derivatives through differentiation. Here we start with the wavelet expansion of the highest derivative and we obtain the wavelet expansion of lower order derivatives and the solution itself through integration.

In this thesis we will adopt the method of Chen and Hsiao method (CHM). Chen and Hsiao demonstrated the possibilities of their method by solving linear system of Ordinary Differential Equations (ODEs) and Partial Differential Equations (PDEs) [23, 24].

The main idea of Chen and Hsiao method (CHM) technique is to convert a differential equation into a system of algebraic equations; which involves a finite number of variables, the solution procedure is either reduced or simplified accordingly[17,18].

The boundary conditions are incorporated by using integration constants. This approach has been realized for the Haar wavelet by Chen and Hsiao method [18, 20].

The procedure is given in the following five steps.

Step (1): In the differential equation. Expand the highest derivative in its Haar series.

Step (2): Integrate the expansion in step (1) repeatedly and using the given conditions until getting the expansion of the solution $u(x)$ this way, we have the expansion of the solution and all of its derivatives that appear in the equation.

Step (3): Substitute the expansion of the solution and its derivatives obtained in step (2) into the equation and evaluate at the collocation points $x_l = \frac{2l-1}{4m}$ or $x_l = \frac{l-0.5}{2m}$, $l = 1,2,\dots,2m$ for a given resolution M ; the result is the system of algebraic equations.

Step (4): Solve the system of algebraic equations obtained in step (3), For the wavelet coefficients a_i.

Step (5): Replace the coefficients a_i in the expansion of the solution to get the solution with resolution M.

5.2 Haar wavelet transformation for linear ordinary differential equations

In this section, we solve second order linear ordinary differential equation using Haar wavelet method.

The general form for a second order linear ODE is

$$u'' + \mu_1(x)u'(x) + \mu_2(x)u(x) = f(x) \tag{5.1}$$

where $x \in [a, b]$

We intend to do J levels of resolutions, hence we let $2M = 2^{J+1}$. The interval $[a, b]$ will be divided into $2M$ subintervals hence $\Delta x = \frac{b-a}{2M}$ and the matrices are of dimensions $2M \times 2M$.

5.2.1 Initial value problems

To explain the approximation of solutions to a second-order linear IVP using Haar wavelet series. Consider the general second-order linear IVP

$$u''(x) + \mu_1(x)u'(x) + \mu_2(x)u(x) = f(x) \tag{5.2}$$

with initial conditions $u(0) = \alpha$ and $u'(0) = \sigma$.

where the coefficients $\mu_1(x)$ and $\mu_2(x)$ are assumed to be expandable in Haar series.

To begin the approximation, we follow Ülo Lepik [15, 21, 22, 24].

Step (1): We assume that

$$u''(x) = \sum_{i=1}^{2M} a_i h_i(x), \text{ for a given resolution level } M \tag{5.3}$$

Step (2): Now integrate (5.3) and (5.4) from 0 to x, to get

$$u'(x) = \sum_{i=1}^{2M} a_i P_{i,1}(x) + u'(0) = \sum_{i=1}^{2M} a_i P_{i,1}(x) + \sigma \tag{5.4}$$

$$u(x) = \sum_{i=1}^{2M} a_i P_{i,2}(x) + x\sigma + u(0)$$

$$= \sum_{i=1}^{2M} a_i P_{i,2}(x) + x\sigma + \alpha \tag{5.5}$$

Step (3): Substitute (5.3)-(5.5) into (5.2), simplify and evaluate at the collocation points $x_l = \frac{2l-1}{4m}, l = 1,2,\dots,2m$.

$$\sum_{i=1}^{2M} a_i [h_i(x_l) + \mu_1(x_l)P_{i,1}(x_l) + \mu_2(x_l)P_{i,2}(x_l)]$$

$$= f(x_l) - \mu_1(x_l)\sigma - \mu_2(x_l)[x_l \sigma - \alpha] \tag{5.6}$$

Step (4): Solve the system in step (3) for the wavelet coefficients a_i.

Step (5): Obtain the numerical solution for $u(x)$ by using the coefficients a_i in the wavelet series expansion of the solution.

The procedure is generalized to higher-order IVPs in similar way

Example 5.1: Consider the second-order homogeneous IVP:

$$u''(x) + \frac{1}{4}u(x) = 0, \quad 0 < x < 1$$

$$u(0) = 1, \qquad u'(0) = 0.$$

with the exact solution

$$u(x) = \cos\left(\frac{x}{2}\right)$$

Using three levels of Haar wavelet $(J = 3)$

Step (1): Expand the highest derivative $u''(x)$,

$$u''(x) = \sum_{i=1}^{2M} a_i \, h_i(x)$$

Step (2): Integrate both sides from 0 to x.

$$\int_0^x u''(\tau) \, d\tau = \int_0^x \sum_{i=1}^{2M} a_i h_i(\tau) d\tau$$

$$u'(x) - u'(0) = \sum_{i=1}^{2M} a_i \int_0^x h_i(\tau) d\tau$$

$$u'(x) = \sum_{i=1}^{2M} a_i P_{i,1}(x)$$

Again, integrate both sides from 0 to x.

$$\int_0^x u'(\tau) d\tau = \int_0^x \sum_{i=1}^{2M} a_i P_{i,1}(\tau) d\tau$$

$$u(x) - u(0) = \sum_{i=1}^{2M} a_i P_{i,2}(x)$$

$$u(x) = \sum_{i=1}^{2M} a_i P_{i,2}(x) + 1$$

Step (3): Substitute $u(x)$ and its derivatives in the equation and evaluate at the collocation points x_l.

$$u''(x) + \frac{1}{4}u(x) = 0$$

$$\sum_{i=1}^{2M} a_i\, h_i(x) + \frac{1}{4}\sum_{i=1}^{2M} a_i P_{i,2}(x) = \frac{-1}{4}$$

$$\sum_{i=1}^{2M} a_i\left[h_i(x_l) + \frac{1}{4}P_{i,2}(x_l)\right] = \frac{-1}{4}\ ,l = 1\,,2\,,\dots,16.$$

Step (4): Solving the system of linear equation by direct method. We obtain the wavelet coefficients a_i.

Step (5): Substitute the coefficients in $u(x)$ to get the solution.

Table 5.1 shows the exact and the approximate solutions at different nodes.

Table 5.1: The numerical solution of example 5.1

Node(x/32)	Exact	Haar solution	Error
1	0.99987793	0.99987799	6.000×10^{-8}
3	0.99890157	0.99890190	3.400×10^{-7}
5	0.99694979	0.99695021	4.100×10^{-7}
7	0.99402452	0.99402576	1.250×10^{-6}
9	0.99012859	0.99012999	1.400×10^{-6}
11	0.98526582	0.98526860	2.780×10^{-6}
13	0.97944095	0.97944397	3.010×10^{-6}
15	0.97265968	0.97266461	4.940×10^{-6}
17	0.96492862	0.96493385	5.240×10^{-6}
19	0.95625532	0.95626301	7.690×10^{-6}
21	0.94664826	0.94665631	8.050×10^{-6}
23	0.93611681	0.93612783	1.101×10^{-5}
25	0.92467126	0.92468269	1.143×10^{-5}
27	0.91232278	0.91233768	1.489×10^{-5}
29	0.89908344	0.89909880	1.536×10^{-5}
31	0.88496616	0.88498545	1.929×10^{-5}

Maximum absolute error of Haar wavelet for IVP is 1.929×10^{-5}
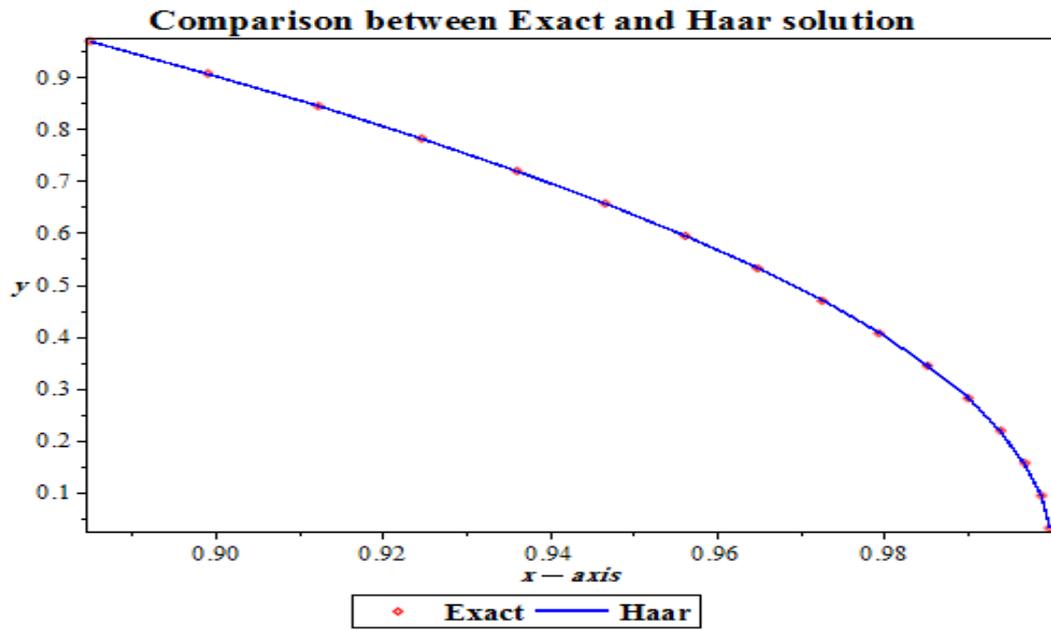
Figure 5.1: The exact and numerical solutions of example 5.1
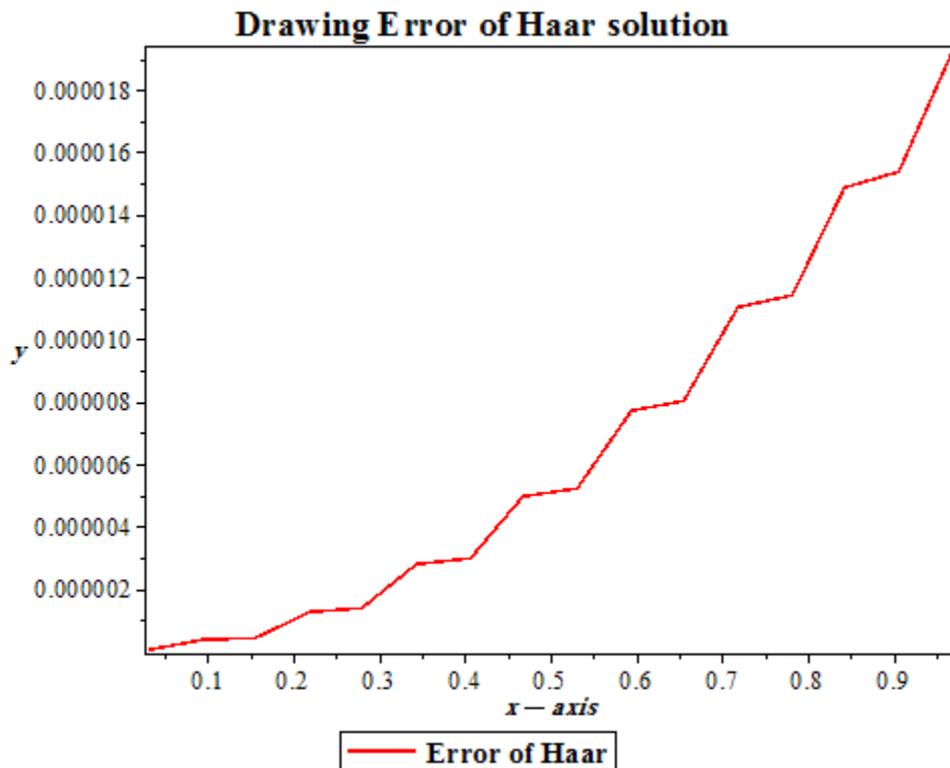
Figure 5.2: Error in the wavelet solution of example 5.1

Example 5.2: Consider the second-order inhomogeneous IVP:

$$u''(x) + u(x) = \sin(x) + x\cos(x), \quad 0 < x < 1$$

$$u(0) = 1, \quad u'(0) = 1.$$

with the exact solution

$$u(x) = \cos(x) + \frac{5}{4}\sin(x) + \frac{1}{4}(x^2 \sin(x) - x\cos(x))$$

Using three levels of Haar wavelet ($J = 3$)

Step (1): Expand the highest derivative $u''(x)$,

$$u''(x) = \sum_{i=1}^{2M} a_i h_i(x)$$

Step (2): Integrate both sides from 0 to x.

$$\int_0^x u''(\tau)\, d\tau = \int_0^x \sum_{i=1}^{2M} a_i h_i(\tau) d\tau$$

$$\Rightarrow u'(x) - u'(0) = \sum_{i=1}^{2M} a_i \int_0^x h_i(\tau) d\tau$$

$$u'(x) = \sum_{i=1}^{2M} a_i P_{i,1}(x) + 1$$

Again, integrate both sides from 0 to x.

$$\int_0^x u'(\tau) d\tau = \int_0^x \sum_{i=1}^{2M} a_i P_{i,1}(\tau) d\tau + \int_0^x d\tau$$

$$u(x) - u(0) = \sum_{i=1}^{2M} a_i P_{i,2}(x) + x$$

$$u(x) = \sum_{i=1}^{2M} a_i P_{i,2}(x) + x + 1$$

Step (3): Substitute $u(x)$ and its derivatives in the equation and evaluate at the collocation points x_l.

$$u''(x) + u(x) = sin(x) + x\,cos(x)$$

$$\sum_{i=1}^{2M} a_i \left[h_i(x_l) + P_{i,2}(x_l) \right] = sin(x_l) + x_l\,cos(x_l) - x_l - 1, l = 1, 2, \dots, 16.$$

Step (4): Solving the system of linear equations in *step (3)*, we obtain the wavelet coefficients a_i.

Step (5): Substitute the coefficient in $u(x)$ to get the solution.

Table 5.2 shows the exact and the approximate solutions at different nodes.

Table 5.2: Haar wavelet and $RK4$ method, for example 5.2

Node(x/32)	Exact	Haar solution	Runge-Kutta 4
1	1.03076684	1.03079305	1.03076684
3	1.08949571	1.08963744	1.08949572
5	1.14469969	1.14489222	1.14469969
7	1.19664324	1.19695329	1.19664325
9	1.24559411	1.24595329	1.24559411
11	1.29181853	1.29229015	1.29181853
13	1.33557659	1.33609320	1.33557659
15	1.37711767	1.37773557	1.37711768
17	1.41667605	1.41733311	1.41667606
19	1.45446674	1.45520769	1.45446675
21	1.49068148	1.49145409	1.49068150
23	1.52548513	1.52631904	1.52548514
25	1.55901222	1.55986889	1.55901223
27	1.59136390	1.59225504	1.59136391
29	1.62260531	1.62350935	1.62260532
31	1.65276320	1.65367175	1.65276321
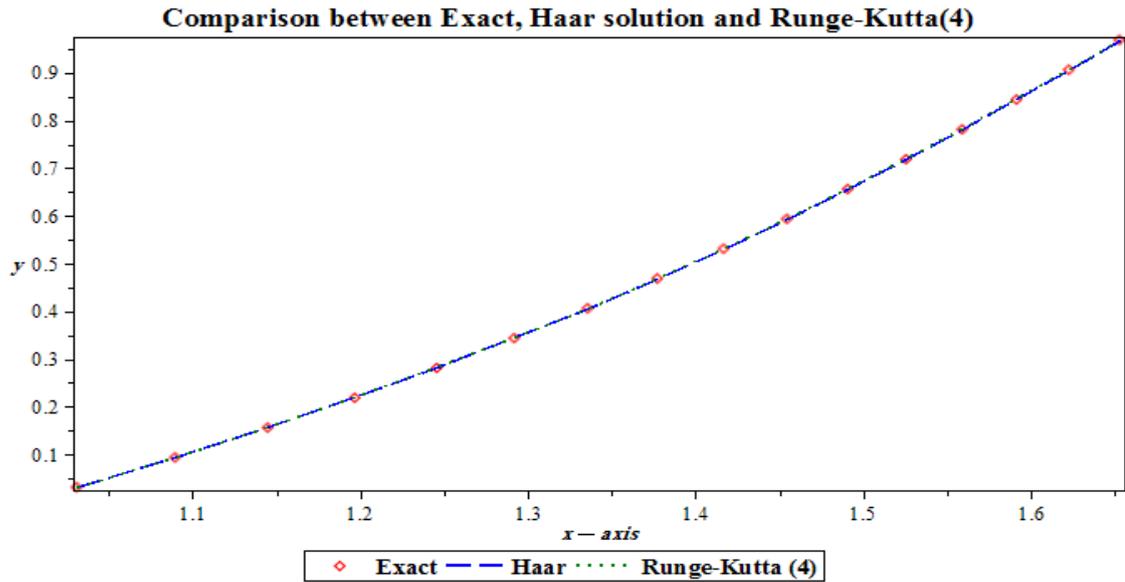
Figure 5.3: The exact and numerical solutions (Haar solution and RK4).

Table 5.3: Error using Haar wavelet method and $RK4$ for example 5.2

Node(x/32)	Error of Haar	Error of RK4
1	2.2610×10^{-5}	1×10^{-8}
3	1.4173×10^{-4}	1×10^{-8}
5	1.9253×10^{-4}	1×10^{-8}
7	3.1004×10^{-4}	1×10^{-8}
9	3.5896×10^{-4}	1×10^{-7}
11	4.7163×10^{-4}	1×10^{-8}
13	5.1662×10^{-4}	1×10^{-8}
15	6.1790×10^{-4}	1×10^{-8}
17	6.5705×10^{-4}	1×10^{-8}
19	7.4095×10^{-4}	1×10^{-8}
21	7.7260×10^{-4}	2×10^{-8}
23	8.3390×10^{-4}	1×10^{-7}
25	8.5663×10^{-4}	1×10^{-8}
27	8.9113×10^{-4}	1×10^{-8}
29	9.0404×10^{-4}	1×10^{-8}
31	9.0855×10^{-4}	1×10^{-8}
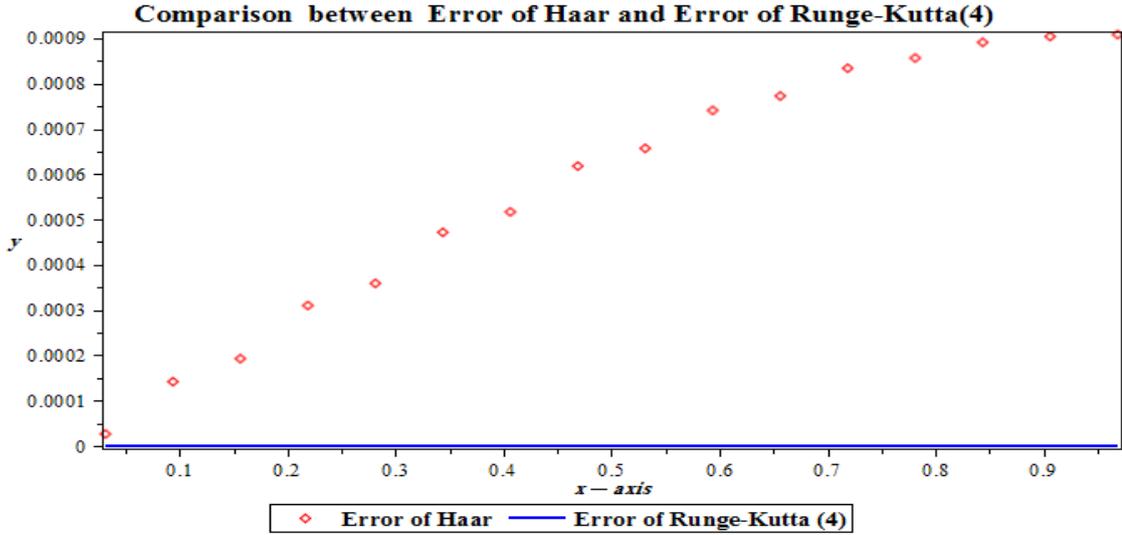
Figure 5.4: Error using Haar wavelet method and $RK4$ for example 5.2

Maximum absolute errors at different resolutions we given the following table.

Table 5.4: Convergence of Haar wavelet for initial value problem.

Level of resolution	$2M$	L_∞	Ratio of L_∞	L_{mre}
$J = 3$	16	9.0855×10^{-4}		5.6024×10^{-4}
$J = 4$	32	2.2735×10^{-4}	0.2525044810	1.3881×10^{-4}
$J = 5$	64	5.6910×10^{-5}	0.2503188916	3.4433×10^{-5}
$J = 6$	128	1.4220×10^{-5}	0.2498682130	8.5655×10^{-6}
$J = 7$	256	3.5600×10^{-6}	0.2503516174	2.1491×10^{-6}
$J = 8$	512	8.9010×10^{-7}	0.2500280899	5.3433×10^{-7}
$J = 9$	1024	2.2011×10^{-7}	0.2472868217	1.3201×10^{-7}
$J = 10$	2048	6.0012×10^{-8}	0.2726454954	3.5901×10^{-8}
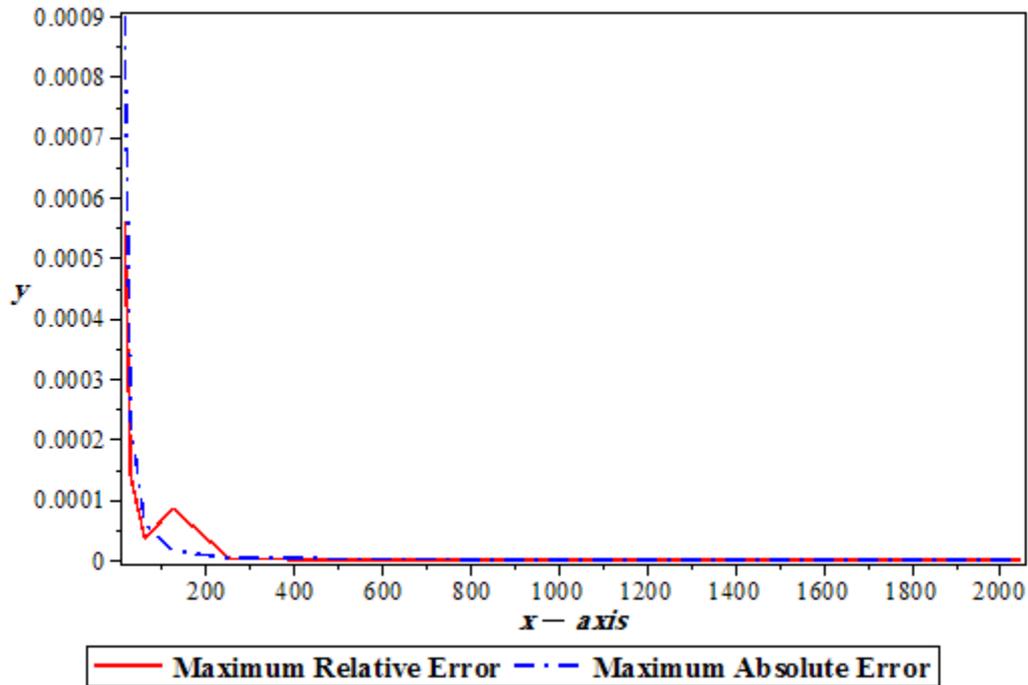
Figure 5.5: Comparison between maximum relative error and maximum absolute error in example 5.2

Example 5.3: Consider the 4th –order of IVP:

$$u^{(4)}(x) + xu(x) = 16\,sin(2x) + xsin(2x), \quad 0 < x < 1.$$

$$u(0) = 0, u'(0) = 2,\ u''(0) = 0, u'''(0) = -8\,.$$

with the exact solution

$$u(x) = sin(2x).$$

Using three levels of Haar wavelet ($J = 3$)

Step (1): Expand the highest derivative $u^{(4)}(x)$,

$$u^{(4)}(x) = \sum_{i=1}^{2M} a_i h_i(x)$$

Step (2): Integrate both sides from 0 to x

$$\int_0^x u^{(4)}(\tau)d\tau = \int_0^x \sum_{i=1}^{2M} a_i h_i(\tau)d\tau$$

$$u'''(x) - u'''(0) = \sum_{i=1}^{2M} a_i P_{i,1}(x) \Longrightarrow u'''(x) = \sum_{i=1}^{2M} a_i P_{i,1}(x) + u'''(0)$$

$$u'''(x) = \sum_{i=1}^{2M} a_i P_{i,1}(x) - 8$$

Again, integrate both sides from 0 to x

$$\int_0^x u'''(\tau)d\tau = \int_0^x \sum_{i=1}^{2M} a_i P_{i,1}(\tau)d\tau - \int_0^x 8\, d\tau$$

$$\Longrightarrow u''(x) - u''(0) = \sum_{i=1}^{2M} a_i P_{i,2}(x) - 8x$$

$$u''(x) = \sum_{i=1}^{2M} a_i P_{i,2}(x) - 8x$$

Again, integrate both sides from 0 to x

$$\int_0^x u''(\tau)d\tau = \int_0^x \sum_{i=1}^{2M} a_i P_{i,2}(\tau)d\tau - \int_0^x 8\tau\, d\tau$$

$$\Longrightarrow u'(x) - u'(0) = \sum_{i=1}^{2M} a_i P_{i,3}(x) - 4x^2$$

$$u'(x) = \sum_{i=1}^{2M} a_i P_{i,3}(x) - 4x^2 + 2$$

Again, integrate both sides from 0 to x.

$$\int_0^x u'(\tau)d\tau = \int_0^x \sum_{i=1}^{2M} a_i P_{i,3}(\tau)d\tau - \int_0^x 4\tau^2\, d\tau + \int_0^x 2\, d\tau$$

$$u(x) - u(0) = \sum_{i=1}^{2M} a_i P_{i,4}(x) - \frac{4}{3}x^3 + 2x$$

$$u(x) = \sum_{i=1}^{2M} a_i P_{i,4}(x) - \frac{4}{3}x^3 + 2x$$

Step (3): Substitute $u(x)$ and its derivatives in the equation and evaluate at the collocation points x_l.

$$u^{(4)}(x) + xu(x) = 16\,sin(2x) + xsin(2x)$$

$$\sum_{i=1}^{2M} a_i\, h_i(x)x \left(\sum_{i=1}^{2M} a_i P_{i,4}(x) - \frac{4}{3}x^3 + 2x \right) = 16sin(2x) + xsin(2x)$$

$$\sum_{i=1}^{2M} a_i \left[h_i(x_l) + x_l P_{i,4}(x_l) \right] = 16sin(2\,x_l) + x_l sin(2x_l) + \frac{4}{3}x_l^4 - 2x_l^2,$$

$$l = 1,2,\dots,16.$$

Step (4): Solving the system of linear equation, we obtain the wavelet coefficients a_i.

Step (5): Substitute the coefficients in $u(x)$ to get the solution.

Table 5.5 shows the exact and the approximate solutions at different nodes.

Table 5.5: The numerical solution of example 5.3

Node(x/32)	Exact	Haar solution	RK4 for system
1	0.06245932	0.06245939	0.06245931
3	0.18640330	0.18640761	0.18640328
5	0.30743851	0.30746207	0.30743864
7	0.42367626	0.42374063	0.42367741
9	0.53330267	0.53343726	0.53330804
11	0.63460708	0.63484409	0.63462502
13	0.72600866	0.72638354	0.72605734
15	0.80608111	0.80662775	0.80619522
17	0.87357494	0.87432576	0.87381471
19	0.92743692	0.92841901	0.92789968
21	0.96682656	0.96806159	0.96766088
23	0.99112919	0.99263072	0.99255173
25	0.99996559	1.00173857	1.00228037
27	0.99319785	0.99523685	0.99681803
29	0.97093160	0.97321914	0.97640324
31	0.93351428	0.93601905	0.94154190
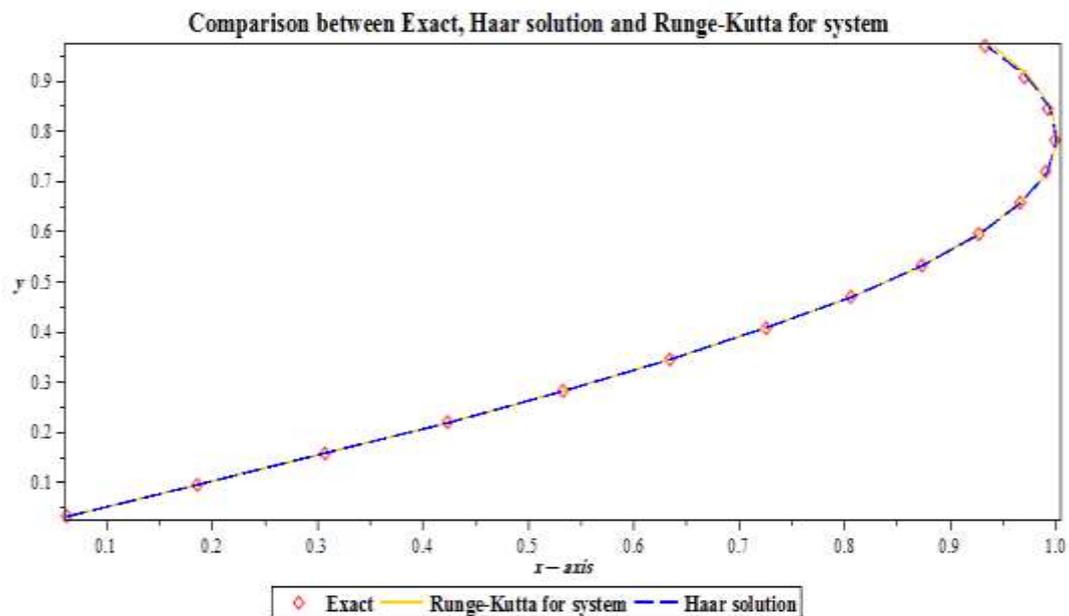
Figure 5.6: The exact and numerical solutions of example 5.3

Table 5.6: Error using Haar wavelet and RK4 for system for example 5.3

Node(x/32)	Error of Haar	Error of RK4 for system
1	0.00000007	0.00000001
3	0.00000431	0.00000002
5	0.00002356	0.00000013
7	0.00006437	0.00000115
9	0.00013459	0.00000537
11	0.00023701	0.00001794
13	0.00037488	0.00004868
15	0.00054664	0.00011411
17	0.00075082	0.00023977
19	0.00098210	0.00046276
21	0.00123504	0.00083432
23	0.00150153	0.00142254
25	0.00177298	0.00231478
27	0.00203900	0.00362018
29	0.00228755	0.00547162
31	0.00250477	0.00802762
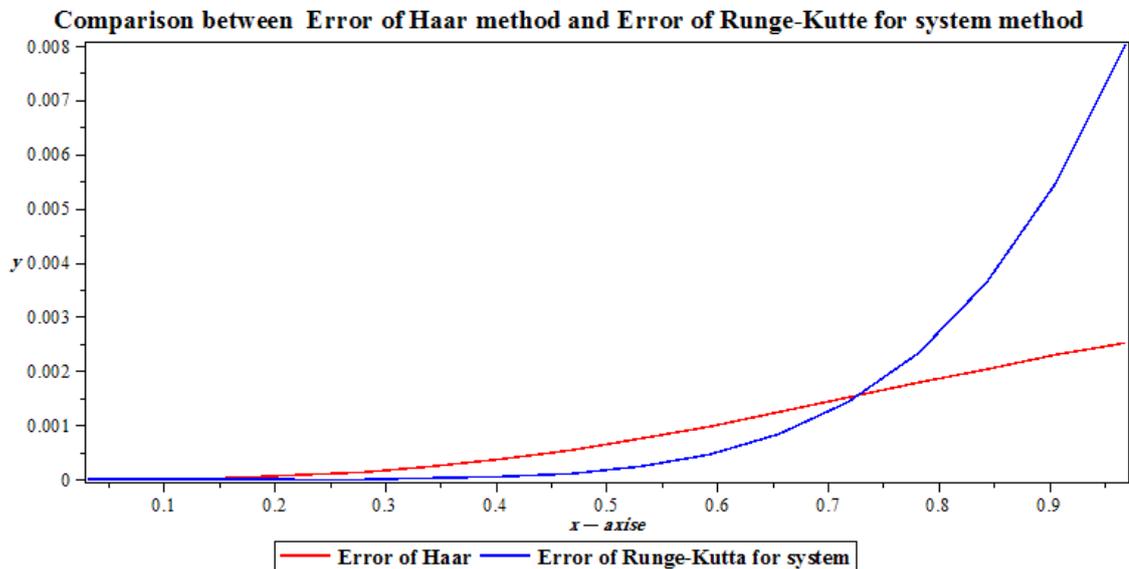
Figure 5.7: Error using Haar wavelet and RK4 for system for example 5.3

Maximum absolute error of Haar wavelet for IVP is 0.00250477

Maximum absolute error of RK4 for IVP is 0.00802762

5.2.2 The general solution for boundary value problems

The objective of this section is to construct a simple collocation method with the Haar basis functions for the numerical solution of linear second-order BVPs arising in the mathematical modeling of different engineering applications. To test applicability of the Haar wavelets, we focus on the following type of BVPs defined in the interval $[0, 1]$:

The general form of a 2nd – order linear BVP is

$$u''(x) + \mu_1(x)u'(x) + \mu_2(x)u(x) = f(x) \qquad 0 < x < 1 \qquad (5.7)$$

subject to one of the following four sets of different boundary conditions.

Case 1: $u'(0) = \alpha_1, \quad u'(1) = \beta_1;$

Case 2: $u(0) = \alpha_2, \quad u(1) = \beta_2;$

Case 3: $u'(0) = \alpha_3, \quad u(1) = \beta_3;$

Case 4: $u(0) = \alpha_4, \quad u'(1) = \beta_4;$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4$ are real constants.

Following Ülo Lepik [15, 21, 24,27]

We introduce the following notation:

$$C_{i,1} = \int_0^1 P_{i,1}(\tau)d\tau$$

Case 1: $u''(x) = \phi(x, u(x), u'(x)) \quad with \quad u'(0) = \alpha_1, \quad u'(1) = \beta_1;$

Step (1): We assume that

$$u''(x) = \sum_{i=1}^{2M} a_i h_i(x) \tag{5.8}$$

Step (2): Now integrate (5.8) from 0 to x.

$$u'(x) = \sum_{i=1}^{2M} a_i P_{i,1}(x) + u'(0) = \sum_{i=1}^{2M} a_i P_{i,1}(x) + \alpha_1 \tag{5.9}$$

Now integrate (5.8) from x to 1.

$$\int_x^1 u''(\tau)d\tau = \int_x^1 \sum_{i=1}^{2M} a_i h_i(\tau)\, d\tau$$

$$\int_x^1 u''(\tau)d\tau = \sum_{i=1}^{2M} [\int_0^1 a_i h_i(\tau)\, d\tau - \int_0^x a_i h_i(\tau)d\tau]$$

$$\beta_1 - u'(x) = a_1 - \sum_{i=1}^{2M} a_i P_{i,1}(x) \tag{5.10}$$

From (5.9) and (5.10), we have $a_1 = \beta_1 - \alpha_1$

Hence the corresponding approximations are

$$u''(x) = (\beta_1 - \alpha_1)h_1(x) + \sum_{i=2}^{2M} a_i h_i(x) \tag{5.11}$$

Integrate (5.11) from 0 to x.

$$u'(x) = \alpha_1 + (\beta_1 - \alpha_1)P_{1,1}(x) + \sum_{i=2}^{2M} a_i P_{i,1}(x) \tag{5.12}$$

Again, integrate (5.12) from 0 to x.

$$u(x) = u(0) + \alpha_1 x + (\beta_1 - \alpha_1)P_{1,2}(x) + \sum_{i=2}^{2M} a_i P_{i,2}(x) \tag{5.13}$$

Step (3): Substituting the value of $u(x), u'(x)$ and $u''(x)$ in the given differential equation, we obtain system of equations, simplify and evaluate at the collocation points $x_l = \frac{2l-1}{4m}, l = 1,2, \dots, 2m$

$$(\beta_1 - \alpha_1)h_1(x_l) + \sum_{i=2}^{2M} a_i h_i(x_l) = \phi(x_l, \alpha_1 + (\beta_1 - \alpha_1)P_{1,1}(x_l) +$$

$$\sum_{i=2}^{2M} a_i P_{i,1}(x_l), u(0) + \alpha_1 x_l + (\beta_1 - \alpha_1)P_{1,2}(x_l) + \sum_{i=2}^{2M} a_i P_{i,2}(x_l)) , l = $$

$$1,2, \dots, 2m$$

Step (4): Solve the above system of equations for the unknowns $u(0)$ and a_i, for $i \neq 1$,

Step (5): Obtain the numerical solution for $u(x)$ in (5.13)

Case 2: $u''(x) = \phi(x, u(x), u'(x))$ $with$ $u(0) = \alpha_2,$ $u(1) = \beta_2$;

Step (1): We assume that

$$u''(x) = \sum_{i=1}^{2M} a_i h_i(x) \tag{5.14}$$

Step (2): Now integrate (5.14) from 0 to x

$$u'(x) = u'(0) + \sum_{i=1}^{2M} a_i P_{i,1}(x) \tag{5.15}$$

Again, integrate (5.15) from 0 to x

$$u(x) = u(0) + xu'(0) + \sum_{i=1}^{2M} a_i P_{i,2}(x) \tag{5.16}$$

Now substituting $x = 1$, in equation (5.16) to get

$$u(1) = u(0) + u'(0) + \sum_{i=1}^{2M} a_i P_{i,2}(1)$$

$$\beta_2 = \alpha_2 + u'(0) + \sum_{i=1}^{2M} a_i P_{i,2}(1)$$

$$u'(0) = \beta_2 - \alpha_2 - \sum_{i=1}^{2M} a_i P_{i,2}(1)$$

$$u(x) = \alpha_2 + x(\beta_2 - \alpha_2 - \sum_{i=1}^{2M} a_i P_{i,2}(1)) + \sum_{i=1}^{2M} a_i P_{i,2}(x)$$

$$u(x) = \alpha_2 + x\beta_2 - x\alpha_2 - x\sum_{i=1}^{2M} a_i P_{i,2}(1) + \sum_{i=1}^{2M} a_i P_{i,2}(x) \tag{5.17}$$

$$u'(x) = \beta_2 - \alpha_2 - \sum_{i=1}^{2M} a_i P_{i,2}(1) + \sum_{i=1}^{2M} a_i P_{i,2}(x) \tag{5.18}$$

Step (3): Substitute these value of $u(x), u'(x)$ *and* $u''(x)$ in the given differential equation we obtain system of equations, simplify and evaluate at the collocation points $x_l = \frac{2l-1}{4m}, l = 1,2,\dots,2m$

$$\sum_{i=1}^{2M} a_i h_i(x_l) = \phi(x_l, \alpha_2 + x_l \beta_2 - x_l \sum_{i=1}^{2M} a_i P_{i,2}(1) +$$

$$\sum_{i=1}^{2M} a_i P_{i,2}(x_l), \beta_2 - \alpha_2 - \sum_{i=1}^{2M} a_i P_{i,2}(1) + \sum_{i=1}^{2M} a_i P_{i,2}(x_l) \,), l =$$

$$1,2,\dots,16 \qquad\qquad (5.19)$$

Step (4): Solve the above system of equations for unknowns a_i,

Step (5): Obtain the numerical solution for $u(x)$ in (5.17).

Case 3: $u''(x) = \phi(x, u(x), u'(x))$ *with* $u'(0) = \alpha_3, \ u(1) = \beta_3;$

Step (1): We assume that

$$u''(x) = \sum_{i=1}^{2M} a_i h_i(x) \tag{5.20}$$

Step (2): Now integrate (5.20) from 0 to x.

$$u'(x) = u'(0) + \sum_{i=1}^{2M} a_i P_{i,1}(x)$$

$$u'(x) = \alpha_3 + \sum_{i=1}^{2M} a_i P_{i,1}(x) \tag{5.21}$$

Now integrate (5.21) from x to 1.

$$u(1) - u(x) = \alpha_3 \int_x^1 \tau d\tau - \sum_{i=1}^{2M} [a_i \int_0^1 P_{i,1}(\tau) d\tau + a_i \int_0^x P_{i,1}(\tau) d\tau]$$

$$u(1) - u(x) = \alpha_3 (1 - x) - \sum_{i=1}^{2M} a_i C_{i,1} + \sum_{i=1}^{2M} a_i P_{i,2}(x)$$

$$u(x) = \beta_3 - \alpha_3 (1 - x) - \sum_{i=1}^{2M} a_i C_{i,1} + \sum_{i=1}^{2M} a_i P_{i,2}(x) \tag{5.22}$$

Step (3): Substituting the values of $u(x), u'(x)$ *and* $u''(x)$ in the given differential equation, we obtain the system of equations, simplify and evaluate at the collocation points $x_l = \frac{2l-1}{4m}, l = 1,2,\dots,2m$

$$\sum_{i=1}^{2M} a_i h_i(x_l) = \phi(x_l, \beta_3 - \alpha_3 (1 - x_l)$$

$$- \sum_{i=1}^{2M} a_i C_{i,1} + \sum_{i=1}^{2M} a_i P_{i,2}(x_l), \alpha_3 + \sum_{i=1}^{2M} a_i P_{i,1}(x_l))$$

Step (4): Solve the above system of equations for the unknowns a_i,

Step (5): Obtain the numerical solution for $u(x)$ in (5.22)

Case 4: $u''(x) = \phi(x, u(x), u'(x))$ with $u(0) = \alpha_4$, $u'(1) = \beta_4$;

Step (1): We assume that

$$u''(x) = \sum_{i=1}^{2M} a_i h_i(x) \qquad (5.23)$$

Step (2): Now integrate (5.23) with respect to τ from x to 1

$$u'(1) - u'(x) = \sum_{i=1}^{2M} a_i \int_0^1 h_i(\tau) d\tau - \sum_{i=1}^{2M} a_i \int_0^x h_i(\tau) d\tau$$

$$u'(x) = u'(1) - a_1 + \sum_{i=1}^{2M} a_i P_{i,1}(x)$$

$$u'(x) = \beta_4 - a_1 + \sum_{i=1}^{2M} a_i P_{i,1}(x) \qquad (5.24)$$

Integrate equation (5.24) with respect to τ from 0 to x.

$$u(x) = u(0) + (\beta_4 - a_1)x + \sum_{i=1}^{2M} a_i P_{i,2}(x)$$

$$u(x) = \alpha_4 + (\beta_4 - a_1)x + \sum_{i=1}^{2M} a_i P_{i,2}(x) \qquad (5.25)$$

Step (3): Substituting these value of $u(x), u'(x)$ and $u''(x)$ in the given differential equation, we obtain system of equations, simplify and evaluate at the collocation points $x_l = \frac{2l-1}{4m}, l = 1, 2, \dots, 2m$

$$\sum_{i=1}^{2M} a_i h_i(x_l) = \phi(x_l, \alpha_4 + (\beta_4 - a_1)x_l + \sum_{i=1}^{2M} a_i P_{i,2}(x_l), \beta_4 - a_1$$

$$+ \sum_{i=1}^{2M} a_i P_{i,1}(x_l))$$

Step (4): Solve the above system of equations for the unknowns a_i, using collocation method.

Step (5): Obtain the numerical solution for $u(x)$ in (5.25)

Example 5.4: Consider the second-order homogeneous the BVPs :

$$u''(x) - 5u'(x) = 0, \quad 0 < x < 1.$$

$$u(0) = 1, u(1) = 0$$

with the exact solution

$$u(x) = \frac{e^5 - e^{5x}}{e^5 - 1}$$

Using three levels of Haar wavelet ($J = 3$)

Step(1): Expand $u''(x)$ in wavelet series.

$$u''(x) = \sum_{i=1}^{2M} a_i h_i(x)$$

Step (2): Integrate both sides from 0 to x.

$$\int_0^x u''(\tau) \, d\tau = \int_0^x \sum_{i=1}^{2M} a_i h_i(\tau) d\tau$$

$$u'(x) - u'(0) = \sum_{i=1}^{2M} a_i \int_0^x h_i(\tau) d\tau$$

$$u'(x) = u'(0) + \sum_{i=1}^{2M} a_i P_{i,1}(x)$$

Again, integrate both sides from 0 to x

$$u(x) = u(0) + xu'(0) + \sum_{i=1}^{2M} a_i P_{i,2}(x)$$

Now substitute $u(1)$ to determine $u'(0)$

$$u(1) = u(0) + u'(0) + \sum_{i=1}^{2M} a_i P_{i,2}(1)$$

$$u'(0) = -1 - \sum_{i=1}^{2M} a_i P_{i,2}(1)$$

$$u(x) = u(0) - x - x \sum_{i=1}^{2M} a_i P_{i,2}(1) + \sum_{i=1}^{2M} a_i P_{i,2}(x)$$

$$u(x) = 1 - x - x \sum_{i=1}^{2M} a_i P_{i,2}(1) + \sum_{i=1}^{2M} a_i P_{i,2}(x)$$

Step (3): Substitute $u(x)$ and its derivatives in the equation, and evaluate at the collocation points x_l.

$$u''(x) - 5u'(x) = 0$$

$$\sum_{i=1}^{2M} a_i \left[h_i(x_l) + 5P_{i,2}(1) - 5P_{i,1}(x_l) \right] = -5, \quad l = 1,2,\dots,16$$

Step (4): Evaluate at the collocation points x_l, solve the system of the linear equations for wavelet coefficients a_i.

Step (5): Substitute the coefficient in $u(x)$ to get the solution.

Table 5.7 shows the exact and the approximate solutions at different nodes.

Table 5.7: The numerical solution of example 5.4

Node(x/32)	Exact	Haar Solution	Error
1	0.99885276	0.99885507	0.00000231
3	0.99594341	0.99595121	0.00000780
5	0.99196679	0.99198147	0.00001468
7	0.98653140	0.98655462	0.00002322
9	0.97910211	0.97913582	0.00003371
11	0.96894747	0.96899391	0.00004644
13	0.95506772	0.95512935	0.00006163
15	0.93609635	0.93617573	0.00007938
17	0.91016556	0.91026508	0.00009952
19	0.87422238	0.87484378	0.00012141
21	0.82627729	0.82642091	0.00014362
23	0.76006071	0.76022417	0.00016346
25	0.66955337	0.66972955	0.00017617
27	0.54584451	0.54601839	0.00017388
29	0.37675454	0.37689837	0.00014383
31	0.14563596	0.14570195	0.00006598
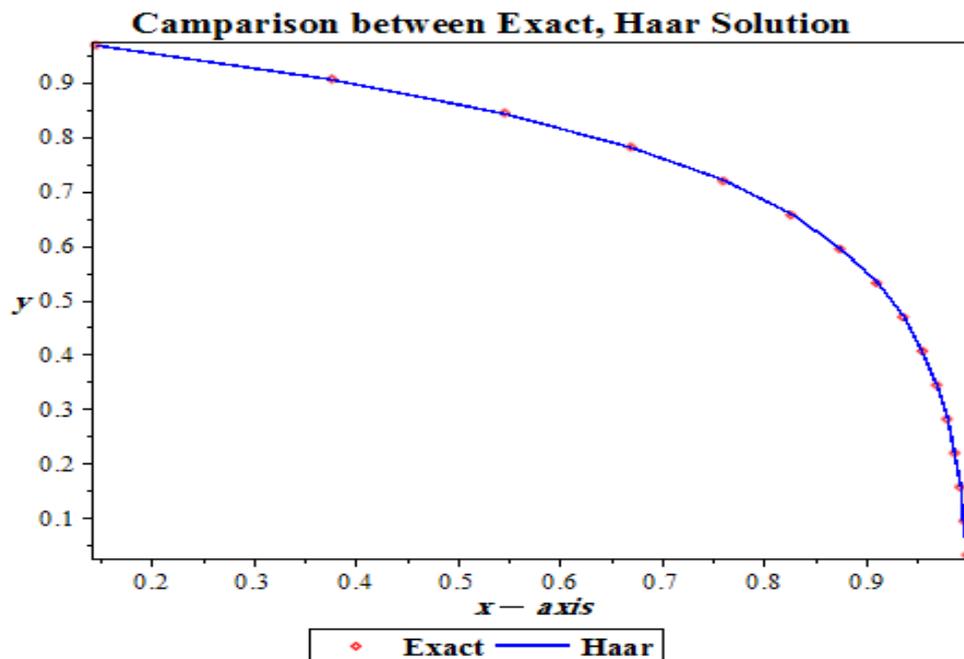
Figure 5.8: The exact and numerical solutions of example 5.4
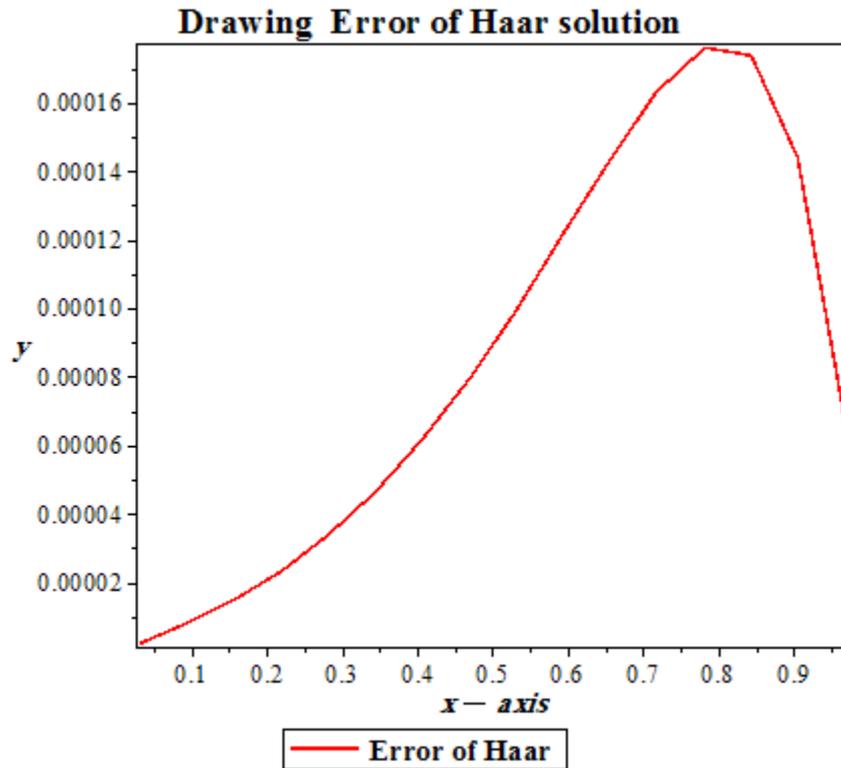
Figure 5.9: Error in example 5.4

Maximum absolute errors at different resolutions we given the following table.

Table 5.8: Convergence of Haar wavelet for boundary value problem.

Level of resolution	$2M$	L_∞	Ratio of L_∞
$J = 3$	16	1.7617×10^{-4}	
$J = 4$	32	4.3332×10^{-5}	0.2459687100
$J = 5$	64	1.0517×10^{-5}	0.2427109084
$J = 6$	128	6.4961×10^{-6}	0.2545008535
$J = 7$	256	1.6532×10^{-6}	0.2501550082
$J = 8$	512	4.1357×10^{-7}	0.2500901309
$J = 9$	1024	1.0343×10^{-7}	0.2495810433
$J = 10$	2048	2.5814×10^{-8}	0.2491245803
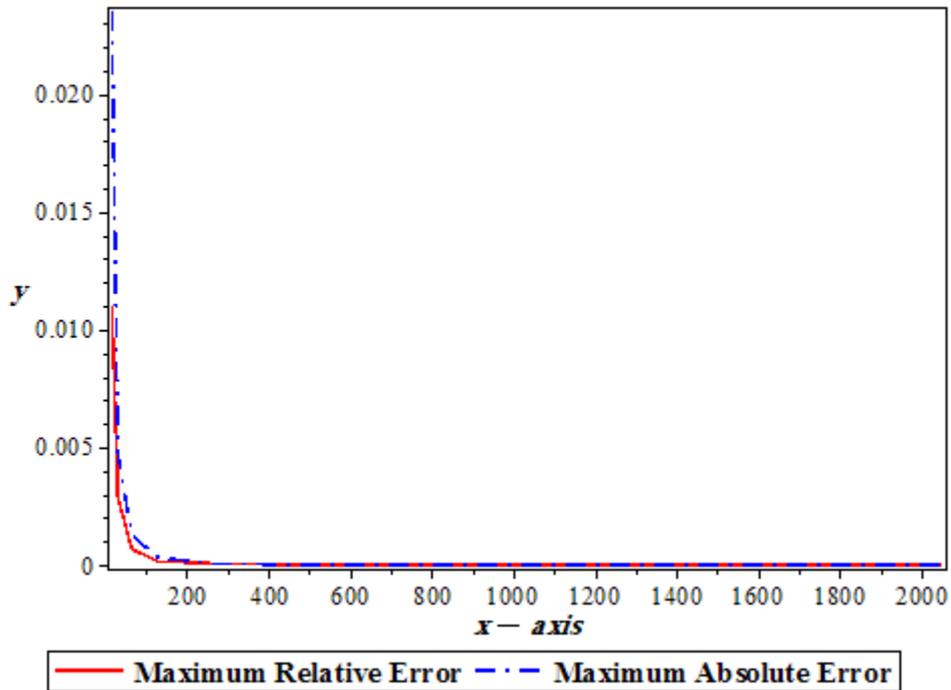
Figure 5.10: Comparison between maximum relative error and maximum absolute error in example 5.4

Example 5.5: Consider the second–order inhomogeneous the BVP:

$$u''(x) + u(x) = \sin(x) + x\cos(x), \quad 0 < x < 1.$$

$$u(0) = 1, u(1) = 1.667433$$

with the exact solution

$$u(x) = \cos(x) + \frac{5}{4}\sin(x) + \frac{1}{4}(x^2\sin(x) - x\cos(x))$$

Using three levels of Haar wavelet ($J = 3$)

Step (1): Expand $u''(x)$ in wavelet series

$$u''(x) = \sum_{i=1}^{2M} a_i\, h_i(x)$$

Step (2): Integrate both sides from 0 to x

$$\int_0^x u''(\tau)\, d\tau = \int_0^x \sum_{i=1}^{2M} a_i h_i(\tau)\, d\tau$$

$$u'(x) - u'(0) = \sum_{i=1}^{2M} a_i P_{i,1}(x)$$

$$u'(x) = u'(0) + \sum_{i=1}^{2M} a_i P_{i,1}(x)$$

Again, integrate both sides from 0 to x again

$$u(x) - u(0) = \sum_{i=1}^{2M} a_i P_{i,2}(x) + x\, u'(0)$$

$$u(x) = u(0) + xu'(0) + \sum_{i=1}^{2M} a_i P_{i,2}(x)$$

where $u'(0)$ is unknown, $u'(0)$ can be found by considering

$$u(1) = 1.667433$$

$$u(1) = \sum_{i=1}^{2M} a_i P_{i,2} + u'(0) + 1$$

$$u'(0) = 0.667433 - \sum_{i=1}^{2M} a_i P_{i,2}(1)$$

$$u(x) = \sum_{i=1}^{2M} a_i P_{i,2}(x) - x \sum_{i=1}^{2M} a_i P_{i,2}(1) + 0.667433\, x + 1$$

Step (3): Substitute $u(x)$ and its derivatives in the equation

$$u''(x) + u(x) = sin(x) + x\, cos(x)$$

$$\sum_{i=1}^{2M} a_i h_i(x) + \sum_{i=1}^{2M} a_i P_{i,2}(1) + 1 + x \left[-\sum_{i=1}^{2M} a_i P_{i,1}(1) + 0.667433 \right]$$

$$= sin(x) + x \, cos(x)$$

$$\sum_{i=1}^{2M} a_i \left[h_i(x_l) + P_{i,2}(x_l) - x_l P_{i,2}(1) \right]$$

$$= sin(x_l) + x_l \, cos(x_l) - 0.667433 x_l - 1,$$

$$l = 1, 2, \dots, 16$$

Step (4): Evaluate at the collocation points x_l, solve the system of linear equation for wavelet coefficients a_i.

Step (5): Substitute the coefficient in $u(x)$ to get the solution.

Table 5.8 shows the exact and the approximate of Haar wavelet, finite difference and linear shooting solutions at different nodes.

Table 5.9: The numerical solution for example 5.5

Node(x/32)	Exact	Haar solution	Finite Difference	Linear Shooting
1	1.03076684	1.03731194	1.01223017	1.01228100
3	1.08949571	1.10240672	1.03942495	1.03941897
5	1.14469969	1.15966881	1.06995086	1.06994136
7	1.19664324	1.20816745	1.10344680	1.10343418
9	1.24559411	1.25667542	1.13957272	1.13955791
11	1.29181853	1.29703658	1.17801204	1.17799453
13	1.33557659	1.33667609	1.21847403	1.21845484
15	1.37711767	1.37260730	1.26069621	1.26006759
17	1.41667605	1.41526066	1.30444677	1.30442598
19	1.45446674	1.45050873	1.34952690	1.34950727
21	1.49068148	1.48339958	1.39577316	1.39575340
23	1.52548513	1.53448845	1.44305974	1.44309160
25	1.55901222	1.54951312	1.49130075	1.49128505
27	1.59136390	1.58348080	1.54045242	1.54044062
29	1.62260531	1.61703899	1.59051518	1.59050700
31	1.65276320	1.65066385	1.64153577	1.64153278
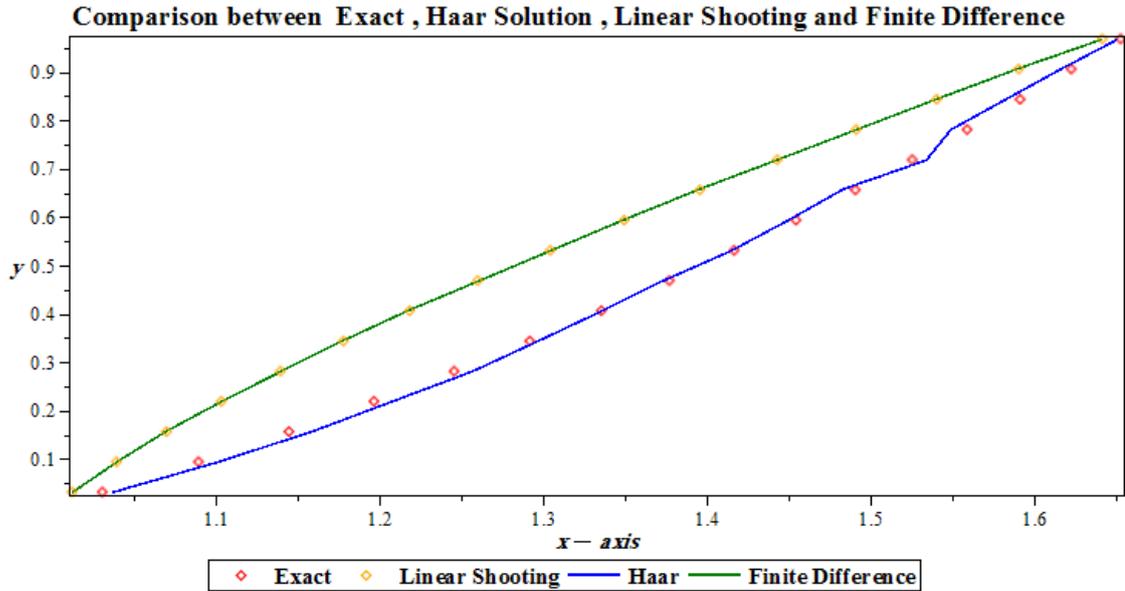
Figure 5.11: The exact and numerical solutions(Haar solution, linear shooting and

finite difference)

Table 5.10: Error in example 5.5

Node(x/32)	Error of Haar	Error of Finite Difference	Error of Linear Shooting
1	0.00654510	0.01853667	0.01853874
3	0.01291101	0.05007075	0.05007674
5	0.01496912	0.07474883	0.07475833
7	0.01152421	0.09319644	0.09320907
9	0.01108131	0.15602139	0.1563620
11	0.00521805	0.21380649	0.21382400
13	0.00109950	0.11710256	0.11712175
15	0.00451037	0.11642146	0.11644175
17	0.00141539	0.11222928	0.11225007
19	0.00395801	0.10503984	0.10496046
21	0.00728190	0.09490832	0.09492809
23	0.00900332	0.08242539	0.08244353
25	0.00949910	0.06771147	0.06772716
27	0.00788310	0.05091148	0.05092388
29	0.00556632	0.03207013	0.03209831
31	0.00209935	0.01122743	0.01123041
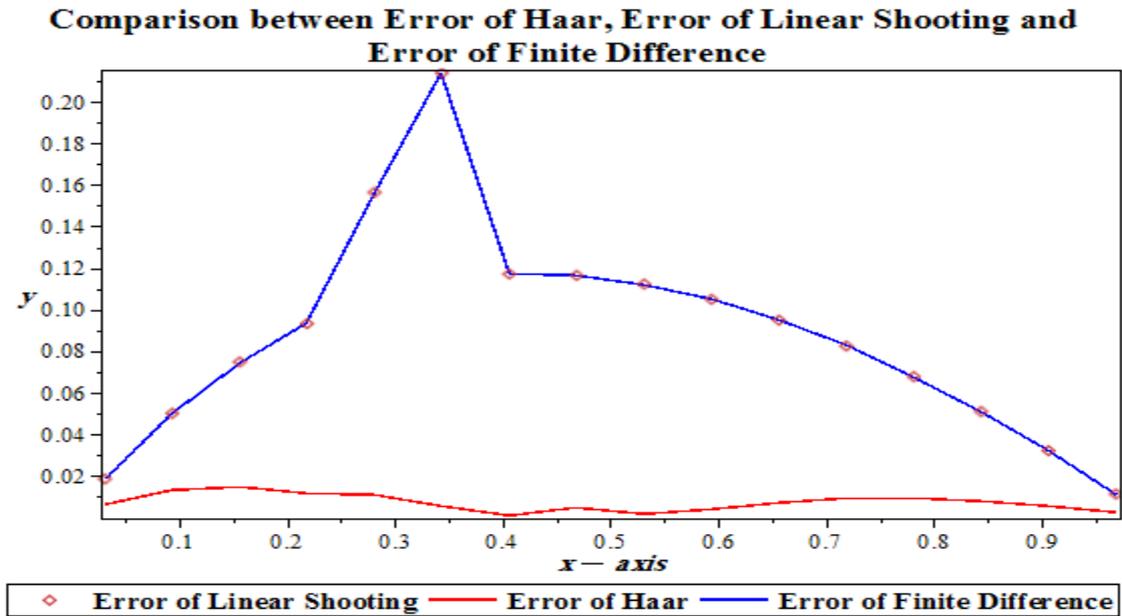
Figure 5.12: Error in example 5.5

Table 5.11: Maximum absolute error of the equation

Maximum absolute error		
Haar Solution	Finite Difference	Linear Shooting
0.01496912	0.21380649	0.21382400

5.3 Haar wavelet method for linear partial differential equations

In this section, we investigate the method of Haar Wavelet for solving partial differential equation. Our investigation is a computational one, so we use two model problems; the well-known Klein-Gordon equation and a wave-like equation. Both are second-order linear homogeneous in one dimension. We will use the notation $\dot{u} = u_t$, $\dot{u}' = u_{tx}$, and so on.

This method consists of reducing the problem into a system of algebraic equations by first expanding the terms of maximum derivative in the equation as Haar wavelet series with a finite number of terms. Second, we

use the operational matrix of Haar wavelet which we established earlier, then the system of algebraic equations for the wavelet coefficient

Example 5.6: Consider the linear Klein-Gordon equation [17]

$$u_{tt} - u_{xx} = u, \qquad 0 < x < 1 \tag{5.26}$$

subject to the initial conditions

$$u(x, 0) = 1 + sin(x), \quad u_t(x, 0) = 0. \tag{5.27}$$

and the boundary conditions

$$u(0, t) = u(1, t) = 0, \qquad t > 0. \tag{5.28}$$

with exact solution

$$u(x, t) = sin(x) + cosh(t)$$

Step (1): Using resolution level $J = 4$

Let $t_s = \frac{(s-1)}{N}, s = 1, 2, \dots, N$, and let $x_l = \frac{2l-1}{4m}, l = 1, 2, \dots, 2m$. Now, expand $\ddot{u}''(x, t)$ in terms of Haar wavelets

$$\ddot{u}''(x, t) = \sum_{i=1}^{2M} a_i h_i(x), \qquad t \in (t_s, t_{s+1}], \tag{5.29}$$

where the column vector a_i is constant in the subinterval $t \in (t_s, t_{s+1}]$.

Step (2): Integrate equation.(5.30) twice with respect to t from t_s to t and twice from 0 to x,

$$\dot{u}''(x, t) = (t - t_s) \sum_{i=1}^{2M} a_i h_i(x) + \dot{u}''(x, t_s), \tag{5.30}$$

$$u''(x,t) = \frac{(t-t_s)^2}{2} \sum_{i=1}^{2M} a_i h_i(x) + (t-t_s)\dot{u}''(x,t_s) + u''(x,t_s) \quad (5.31)$$

$$\ddot{u}'(x,t) = \sum_{i=1}^{2M} a_i P_{i,1}(x) + \ddot{u}'(0,t) \quad (5.32)$$

$$\ddot{u}(x,t) = \sum_{i=1}^{2M} a_i P_{i,2}(x) + \ddot{u}(0,t) + x\ddot{u}'(0,t) \quad (5.33)$$

$$\dot{u}(x,t) = (t-t_s)\sum_{i=1}^{2M} a_i P_{i,2}(x) + \dot{u}(x,t_s) + \dot{u}(0,t) - \dot{u}(0,t_s)$$

$$+x[\dot{u}'(0,t) - \dot{u}'(0,t_s)] \quad (5.34)$$

$$u(x,t) = \frac{(t-t_s)^2}{2} \sum_{i=1}^{2M} a_i P_{i,2}(x) + u(x,t_s) + (t-t_s)\dot{u}(x,t_s) + u(0,t)$$

$$-u(0,t_s) - (t-t_s)\dot{u}(0,t_s) + x[u'(x,t_s) - u'(0,t_s) -$$

$$(t-t_s)\dot{u}'(0,t_s) \quad (5.35)$$

Using the boundary conditions, equations.(5.33)-(5.35) become

$$\ddot{u}(x,t) = \sum_{i=1}^{2M} a_i P_{i,2}(x) - x\sum_{i=1}^{2M} a_i P_{i,2}(1) \quad (5.36)$$

$$\dot{u}(x,t) = (t-t_s)\sum_{i=1}^{2M} a_i P_{i,2}(x) + \dot{u}(x,t_s)$$

$$-x(t-t_s)\sum_{i=1}^{2M} a_i P_{i,2}(1) \quad (5.37)$$

$$u(x,t) = \frac{(t-t_s)^2}{2}\sum_{i=1}^{2M} a_i P_{i,2}(x) + u(x,t_s) + (t-t_s)\dot{u}(x,t_s) -$$

$$\frac{x}{2}(t-t_s)^2 \sum_{i=1}^{2M} a_i P_{i,2}(1) \quad (5.38)$$

Evaluate equations (5.30)-(5.31) and equations (5.35) at $x = x_l$ and $t = t_{s+1}$, using $\Delta t = t_{s+1} - t_s$

$$\dot{u}''(x_l, t_{s+1}) = \Delta t \sum_{i=1}^{2M} a_i \, h_i(x_l) + \dot{u}''(x_l, t_s), \tag{5.39}$$

$$u''(x_l, t_{s+1}) = \frac{\Delta t^2}{2} \sum_{i=1}^{2M} a_i h_i(x_l) + \Delta t \, \dot{u}''(x_l, t_s) + u''(x_l, t_s) \tag{5.40}$$

$$\ddot{u}(x_l, t_{s+1}) = \sum_{i=1}^{2M} a_i P_{i,2}(x_l) - x_l \sum_{i=1}^{2M} a_i P_{i,2}(1) \tag{5.41}$$

$$\dot{u}(x_l, t_{s+1}) = \Delta t \sum_{i=1}^{2M} a_i P_{i,2}(x_l) + \dot{u}(x_l, t_s) + \Delta t \, x_l \sum_{i=1}^{2M} a_i P_{i,2}(1) \tag{5.42}$$

$$u(x_l, t_{s+1}) = \frac{\Delta t^2}{2} \sum_{i=1}^{2M} a_i P_{i,2}(x_l) + u(x_l, t_s) + \Delta t \, \dot{u}(x_l, t_s) -$$

$$\frac{1}{2} \Delta t^2 x_l \sum_{i=1}^{2M} a_i P_{i,2}(1) \tag{5.43}$$

where $x_l = \frac{2l-1}{4m}$, $\qquad l = 1, 2, \dots, 2m.$

Step(3): The scheme is

$$\ddot{u}(x_l, t_{s+1}) - u''(x_l, t_{s+1}) = u(x_l, t_{s+1}) \tag{5.44}$$

$$\text{(i.e.)} \quad \ddot{u}(x_l, t_{s+1}) = u(x_l, t_{s+1}) + u''(x_l, t_{s+1}) \tag{5.45}$$

$$a_i \left[\sum_{i=1}^{2M} a_i P_{i,2}(x_l) \right.$$

$$\left. - x_l \sum_{i=1}^{2M} a_i P_{i,2}(1) - \Delta t^2 \sum_{i=1}^{2M} a_i P_{i,2}(x_l) + \frac{1}{2} \Delta t^2 x_l \sum_{i=1}^{2M} a_i P_{i,2}(1) \right]$$

$$= u(x_l, t_s) + \Delta t [\dot{u}(x_l, t_s) + \dot{u}''(x_l, t_s)]$$

$$+ u''(x_l, t_s) \tag{5.46}$$

Step (4): The Haar coefficients vector a_i are calculated from the system of linear equation. (5.46).

Step (5): The solution of the problem is determined according to equation (5.43) as follows [17].

$$u(x_l, 0) = 1 + sin(x_l)$$

$$u'(x_l, 0) = cos(x_l), \quad u''(x_l, 0) = -sin(x_l), \quad \dot{u}''(x_l, 0) = 0 \quad \text{and}$$
$$\dot{u}(x_l, 0) = 0$$

Adomain decomposition method (ADM)

Adomain decomposition methods are dividing and conquer methods for the parallel and computational solution of partial differential equations of elliptic or parabolic type. They include iterative algorithms for solving the discretized equations, techniques for non-matching grid discretizations and techniques for heterogeneous approximations. An introduction to this method can be found in [26].

The results in table 5.12 and table 5.13 are obtained from [17].

Table 5.12 shows the exact and the approximate at different nodes.

Table 5.12: Numerical result of the linear Klein-Gordon equation

Node	Exact	Haar solution
0.015	2.453×10^{-5}	2.133×10^{-5}
0.046	9.772×10^{-5}	9.385×10^{-5}
0.078	2.525×10^{-4}	2.549×10^{-4}
0.109	1.171×10^{-4}	1.118×10^{-4}
0.453	4.541×10^{-4}	4.556×10^{-4}
0.484	2.112×10^{-4}	2.153×10^{-4}
0.515	3.322×10^{-5}	3.342×10^{-5}

Table 5.13: Maximum absolute error of the equation at different time

t	Maximum absolute error	
	Haar solution	ADM
0.1	5.201×10^{-11}	3.201×10^{-10}
0.3	4.427×10^{-8}	3.121×10^{-7}
0.5	2.325×10^{-5}	1.907×10^{-3}

Example 5.7: Consider the one–dimensional wave-like equation [19]

$$u_{tt} - \frac{x^2}{2} u_{xx} = 0 \quad , 0 < x < 1 , t > 0 \tag{5.47}$$

subject to the initial conditions

$$u(x, 0) = x , \qquad \dot{u}(x, 0) = x^2 \tag{5.48}$$

and boundary conditions

$$u(0, t) = 0, \ u(1, t) = 1 + sinh(t) \quad t > 0 \tag{5.49}$$

the exact solution is

$$u(x, t) = x + x^2 sinh(t)$$

Step (1): Using resolution level $J = 3$

Let $t_s = \frac{(s-1)}{N}, s = 1, 2, \dots, N$, and let $x_l = \frac{2l-1}{4m}, l = 1, 2, \dots, 2m$. Now,

expand $\ddot{u}''(x, t)$ in terms of Haar wavelets

$$\ddot{u}''(x, t) = \sum_{i=1}^{2M} a_i h_i(x) , \qquad t \in [t_s, t_{s+1}], \tag{5.50}$$

Step (2): Integrate equation.(5.50) twice with respect to t from t_s to t and

twice with respect to x from 0 to x, we obtain

$$\dot{u}''(x,t) = (t - t_s) \sum_{i=1}^{2M} a_i h_i(x) + \dot{u}''(x, t_s), \tag{5.51}$$

$$u''(x,t) = \frac{(t - t_s)^2}{2} \sum_{i=1}^{2M} a_i h_i(x) + (t - t_s)\dot{u}''(x, t_s) + u''(x, t_s) \tag{5.52}$$

$$u'(x,t) = \frac{(t - t_s)^2}{2} \sum_{i=1}^{2M} a_i P_i(x) + (t - t_s)[\dot{u}'(x, t_s) - \dot{u}'(0, t_s)]$$

$$+ u'(x, t_s) - u'(0, t_s) + u'(0, t) \tag{5.53}$$

$$u(x,t) = \frac{(t - t_s)^2}{2} \sum_{i=1}^{2M} a_i P_{i,2}(x) + (t - t_s)[\dot{u}(x, t_s) - \dot{u}(0, t_s)$$

$$- x\,\dot{u}'(0, t_s)] + u(x, t) - u(0, t_s) - x[u'(0, t_s) - u'(0, t)]$$

$$+ u(0, t) \tag{5.54}$$

$$\dot{u}(x,t) = (t - t_s)\sum_{i=1}^{2M} a_i P_{i,2}(x) + [\dot{u}(x, t_s) - \dot{u}(0, t_s) + x\,\dot{u}'(0, t_s)]$$

$$+ x\,\dot{u}'(0, t) + \dot{u}(0, t) \tag{5.55}$$

$$\ddot{u}(x,t) = \sum_{i=1}^{2M} a_i P_{i,2}(x) - x\,\ddot{u}'(0, t) + \ddot{u}(0, t) \tag{5.56}$$

Using the initial and boundary conditions, we have the following equations as

$$u(x, 0) = x \,, \dot{u}(x, 0) = x^2 \quad 0 < x < 1$$

$$u(0, t) = g_0(t) = 0$$

$$u(0, t_s) = g_0(t_s) = 0$$

$$u(1, t_s) = g_1(t_s) = 1 + sinh(t_s)$$

$$\dot{u}(0, t_s) = g_0'(t_s) = 0$$

$$\dot{u}(1, t_s) = g_1'(t_s) = cosh(t_s)$$

$$\ddot{u}(0, t_s) = g_0''(t_s) = 0$$

$$\ddot{u}(1, t_s) = g_1''(t_s) = sinh(t_s)$$

Put $x = 1$ in formulas (5.53) and (5.55) and by using condition, we have

$$u'(0, t) - u'(0, t_s) = -\frac{(t - t_s)^2}{2} \sum_{i=1}^{2M} a_i P_{i,2}(1) + (t - t_s)[cosh(t_s)$$

$$-\dot{u}'(0, t_s)] + (1 + sinh(t) - 1 - sinh(t_s)) \qquad (5.57)$$

$$\ddot{u}'(0, t) = -\sum_{i=1}^{2M} a_i P_{i,2}(1) - sinh(t) \qquad (5.58)$$

Substitute equations (5.57) and (5.58) into equations (5.52) through (5.60) and evaluate $x = x_l$ and $t = t_{s+1}$

$$u''(x_l, t_{s+1}) = \frac{(t_{s+1} - t_s)^2}{2} \sum_{i=1}^{2M} a_i h_i(x_l) + (t_{s+1} - t_s)\dot{u}''(x_l, t_s)$$

$$+u''(x_l, t_s) \qquad (5.59$$

$$u'(x_l, t_{s+1}) = \frac{\Delta t^2}{2} \sum_{i=1}^{2M} a_i P_i(x_l) + \Delta t \, \dot{u}'(x_l, t_s) - u'(x_l, t_s) - \frac{\Delta t^2}{2}$$

$$\sum_{i=1}^{2M} a_i P_i(x_l) - \Delta t[\, cosh(t_s)] + sinh(t_{s+1}) - sinh(t_s) \qquad (5.60)$$

$$u(x_l, t_{s+1}) = \frac{\Delta t^2}{2} \sum_{i=1}^{2M} a_i P_{i,2}(x_l) + \Delta t[\dot{u}(x_l, t_s) - \dot{u}(0, t_s)] + u(x_l, t_s)$$

$$-u(0, t_s) - x_l \frac{\Delta t^2}{2} \sum_{i=1}^{2M} a_i P_{i,2}(1) - x_l \Delta t cosh(t_s) - x_l[\, 1 + sinh(t_s)$$

$$- 1 - sinh(t_{s+1})] \qquad (5.61)$$

$$\dot{u}(x_l, t_{s+1}) = \Delta t \sum_{i=1}^{2M} a_i P_{i,2}(x_l) + [\dot{u}(x_l, t_s) - \dot{u}(0, t_s)]$$

$$- \Delta t \, x_l \sum_{i=1}^{2M} a_i P_{i,2}(1) + x_l \cosh(t_s) + x_l \cosh(t_{s+1}) \quad (5.62)$$

$$\ddot{u}(x_l, t_{s+1}) = \sum_{i=1}^{2M} a_i [P_{i,2}(x_l) - x_l P_{i,2}(1)] + x_l \sinh(t_{s+1}) \quad (5.63)$$

$$P_{i,2}(1) = \begin{cases} 0.5 & if \ i = 1 \\ \dfrac{1}{4m^2} & if \ i > 1 \end{cases}$$

Step (3): After substituting value from equation (5.63) in the wave–like equation, we have

$$\sum_{i=1}^{2M} a_i [P_{i,2}(x_l) - x_l P_{i,2}(1)] = \frac{x_l^2}{2} u''(x_l, t_s) + x_l \sinh(t_{s+1}) \quad (5.64)$$

Equation (5.64) is the algebraic from of the wave-like equation (5.47)

Step (4): Solve these algebraic equations for the Haar wavelet coefficients a_i. Then from equation (5.61) we obtain the solution of u, which is very near to the exact solution. This solution process is started with

$$u(x_l, 0) = x_l, \qquad u_t(x_l, 0) = x_l^2, \qquad u'(x_l, 0) = 0, \quad u''(x_l, 0) = 0$$

Step (5): Substitute the coefficient in $u(x, t)$ to get the solution.

Table 5.14: The numerical solution for wave like equation at $t = 0.01$

Node(x/32)	Exact	Haar solution	Finite Difference
1	0.0312597657	0.0312599298	0.0312597648
3	0.0938378920	0.0938380223	0.0938378934
5	0.1564941446	0.1564942398	0.1564941463
7	0.2192285236	0.2192285824	0.2192285265
9	0.2820410288	0.2820410499	0.2820410392
11	0.3449316603	0.3449316424	0.3449316610
13	0.4079004181	0.4079003599	0.4079004237
15	0.4709473022	0.4709472025	0.4709473089
17	0.5340723126	0.5340721700	0.5340723150
19	0.5972754493	0.5972752625	0.5972754320
21	0.6605567124	0.6605564800	0.6605567180
23	0.7239161017	0.7239158225	0.7239161120
25	0.7873536173	0.7873539011	0.7873536280
27	0.8508692592	0.8508688826	0.8508692660
29	0.9144630275	0.9144626001	0.9144630310
31	0.9781349220	0.9781344426	0.9781349360
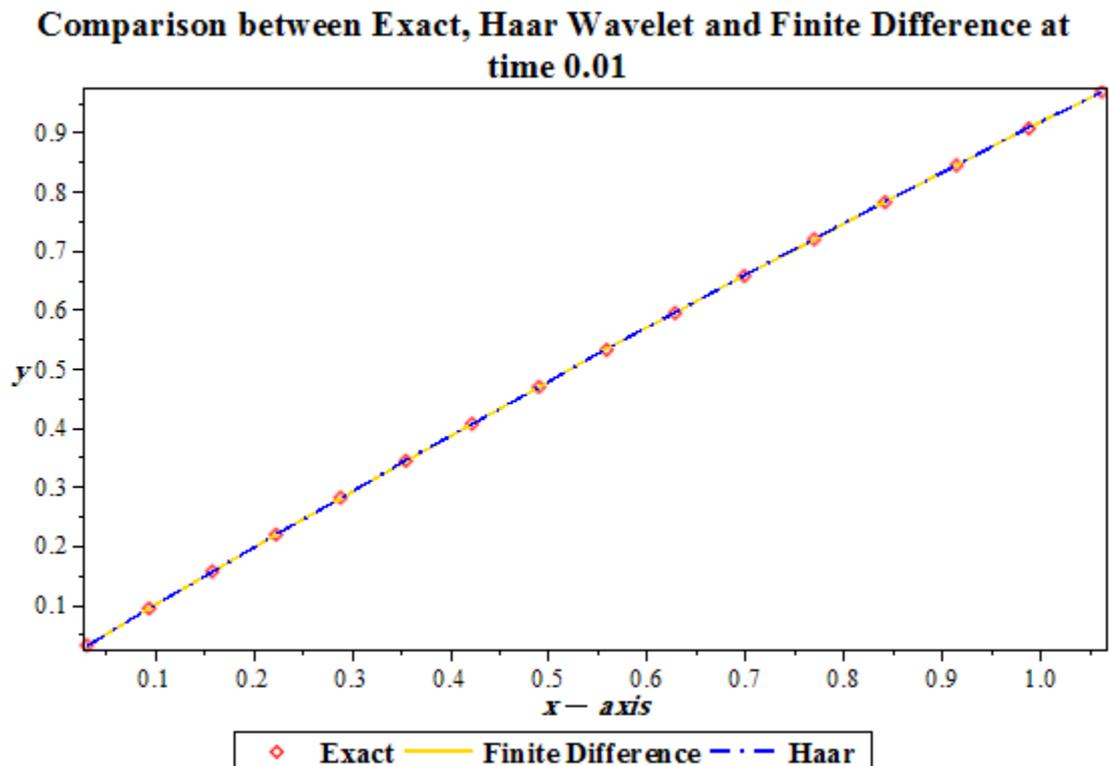
Figure 5.13: The exact and numerical solutions for wave-like equation at $t = 0.01$

Table 5.15: The numerical solution for wave like equation at $t = 0.1$

Node(x/32)	Exact	Haar Solution	Finite Difference
1	0.0313478191	0.0313372164	0.0313483056
3	0.0946303718	0.0945975851	0.0946315015
5	0.1586954773	0.1586392033	0.1586972575
7	0.2235431355	0.2234620805	0.2235455489
9	0.2891733464	0.2890662093	0.2891764013
11	0.3555861101	0.3554515354	0.3555897902
13	0.4227814265	0.4226181026	0.4227857460
15	0.4907592957	0.4905662266	0.4907642376
17	0.5595197175	0.5592956532	0.5595252760
19	0.6290626921	0.6288045412	0.6290688740
21	0.6993882195	0.6990943723	0.6993949970
23	0.7704962996	0.7701758784	0.7705037060
25	0.8423869324	0.8420404232	0.8423949610
27	0.9150601179	0.9146254566	0.9150687420
29	0.9885158562	0.9879813151	0.9885250770
31	1.0627541472	1.0624725665	1.0627620760
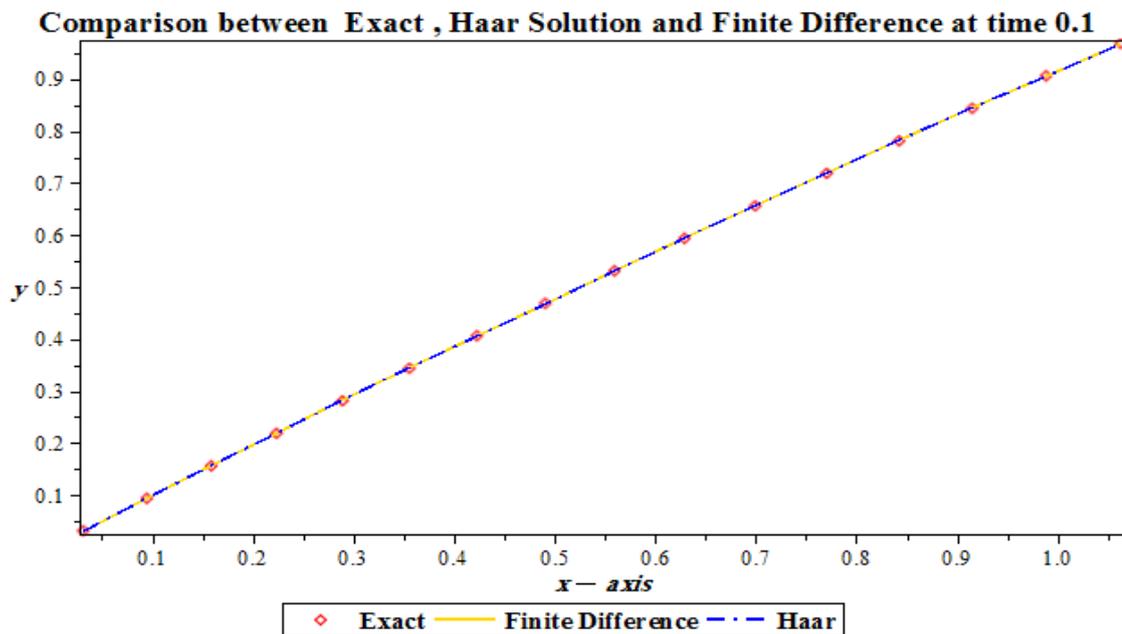
Figure 5.14: The exact and numerical solutions for wave-like equation at $t = 0.1$

Table 5.16: The numerical solution for wave like equation at $t = 0.2$

Node(x/32)	Exact	Haar Solution	Finite Difference
1	0.0314466171	0.0315090527	0.0314505129
3	0.0955195547	0.0957000100	0.0955286338
5	0.1611654297	0.1614547701	0.1611796646
7	0.2283842423	0.2286851414	0.2284035991
9	0.2971759923	0.2975533380	0.2972004527
11	0.3675406799	0.3679767595	0.3675702126
13	0.4394783051	0.4399711571	0.4395128894
15	0.5129888677	0.5135150107	0.5130284782
17	0.5880723679	0.5886319063	0.5881169880
19	0.6647288055	0.6653084730	0.6647783620
21	0.7429581807	0.7436810026	0.7430126860
23	0.8227204935	0.8233571460	0.8228199150
25	0.9041357437	0.9047271867	0.9042000630
27	0.9870839314	0.9870173730	0.9871531000
29	1.0716050567	1.0721586763	1.0716765050
31	1.1576991195	1.1582201252	1.1577405680
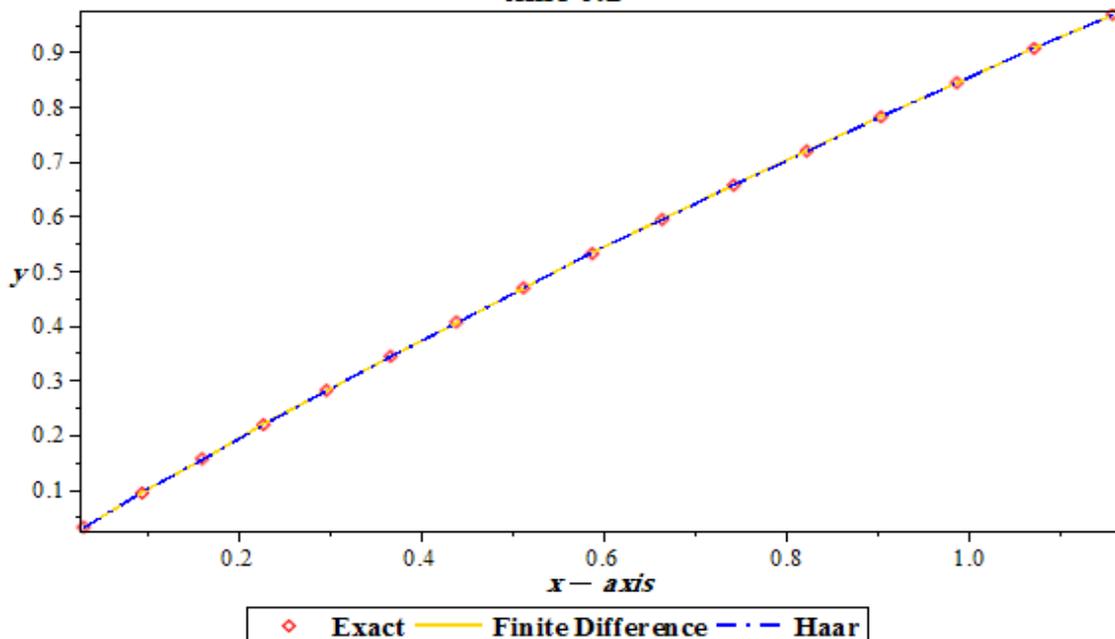
Figure 5.14 : The exact and numerical solutions for wave-like equation at $t = 0.2$

Table 5.17: Maximum error of the equation at different time

t	Maximum absolute error	
	Haar solution	Finite Difference
0.01	4.7934×10^{-7}	1.7400×10^{-8}
0.1	3.4651×10^{-4}	8.0287×10^{-6}
0.2	7.2282×10^{-4}	7.1448×10^{-5}

Example 5.8: Consider the one-dimensional diffusion equation.

$$u_t = u_{xx}, \qquad 0 < x < 1, \qquad t > 0 \qquad (5.65)$$

subject to the initial conditions

$$u(x, 0) = sin(\pi x) \qquad (5.66)$$

and the boundary conditions

$$u(0, t) = u(1, t) = 0, \qquad t > 0 \qquad (5.67)$$

with exact solution

$$u(x, t) = e^{-\pi^2 t} sin(\pi x)$$

Step (1): Assume

Let $x_l = \frac{2l-1}{4m}$, $l = 1, 2, \ldots, 2m$. Now, expand $\ddot{u}''(x, t)$ in terms of Haar wavelets, using time stepping from 0 to .

$$\dot{u}''(x, t) = \sum_{i=1}^{2M} a_i h_i(x), \qquad t \in (0, t] \qquad (5.68)$$

Step (2): Integrate equation (5.62). with respect to t from 0 to t and twice from 0 to x

$$u''(x,t) = t \sum_{i=1}^{2M} a_i h_i(x) - \pi^2 \sin(\pi x) \tag{5.69}$$

$$u'(x,t) = t \sum_{i=1}^{2M} a_i P_{i,1}(x) - \pi \cos(\pi x) + \pi \tag{5.70}$$

$$u(x,t) = t \sum_{i=1}^{2M} a_i P_{i,2}(x) - \sin(\pi x) + \pi x \tag{5.71}$$

Now,

$$\dot{u}'(x,t) = \sum_{i=1}^{2M} a_i P_{i,1}(x) + \dot{u}'(0,t) \tag{5.72}$$

$$\dot{u}(x,t) = \sum_{i=1}^{2M} a_i P_{i,2}(x) + x\dot{u}'(0,t) + \dot{u}(0,t) \tag{5.73}$$

Using the boundary conditions, set $x = 1$ to find $\dot{u}'(0,t)$ in equation (5.73)

$$\dot{u}'(0,t) = -\sum_{i=1}^{2M} a_i P_{i,2}(1) \tag{5.74}$$

$$\dot{u}(x,t) = \sum_{i=1}^{2M} a_i P_{i,2}(x) - x \sum_{i=1}^{2M} a_i P_{i,2}(1) \tag{5.75}$$

Evaluate equations (5.68)-(5.73) at $x = x_l$ and using time stepping.

where $x_l = \frac{2l-1}{4m}$, $\qquad l = 1, 2, \dots, 2m$.

Step(3): The scheme is

$$\dot{u}(x_l,t) = u''(x_l,t) \tag{5.76}$$

$$\sum_{i=1}^{2M} a_i \big[th_i(x_l) - P_{i,2}(x_l) + x_l P_{i,2}(1)\big] = \pi^2 \sin(\pi x_l) \tag{5.77}$$

Step (4): Solve these algebraic equations for the Haar wavelet coefficients a_i. Then from equation (5.77) we obtain the solution of u

Step (5): Substitute the coefficient in $u(x, t)$ to get the solution.

Using resolution level $(J = 3)$ of Haar wavelet.

Table 5.18 shows the exact and the approximate at different nodes.

Table 5.18: The numerical solution for diffusion equation at time $t = 0.1$

Node	X[node]	Exact	Haar	Rel. Error
1	0.06250000	0.00731763	0.00731859	1.309296468007e-04
2	0.12500000	0.01463244	0.01463433	1.287644526372e-04
3	0.18750000	0.02194161	0.02194436	1.251883252893e-04
4	0.25000000	0.02924232	0.02924584	1.202501261167e-04
5	0.31250000	0.03653176	0.03653592	1.140183951630e-04
6	0.37500000	0.04380711	0.04381178	1.065814856472e-04
7	0.43750000	0.05106557	0.05107058	9.804773763998e-05
8	0.50000000	0.05830435	0.05830951	8.854569095200e-05
9	0.56250000	0.06552065	0.06552578	7.822433840886e-05
10	0.62500000	0.07271169	0.07271658	6.725341943953e-05
11	0.68750000	0.07987470	0.07987916	5.582375551514e-05
12	0.75000000	0.08700692	0.08701076	4.414762759099e-05
13	0.81250000	0.09410559	0.09410864	3.245919693256e-05
14	0.87500000	0.10116798	0.10117011	2.101497058229e-05
15	0.93750000	0.10819137	0.10819247	1.009431217516e-05

Maximum Relative Error: 1.309296468007e-04

Using resolution level $(J = 4)$ of Haar wavelet

Table 5.19: The numerical solution for diffusion equation at time $t = 0.1$

Node	X[node]	Exact	Haar	Rel. Error
1	0.03125000	0.00365899	0.00365947	1.312811991005e-04
3	0.09375000	0.01097557	0.01097699	1.298351547165e-04
5	0.15625000	0.01828791	0.01829023	1.269625778754e-04
7	0.21875000	0.02559320	0.02559634	1.227025307863e-04
9	0.28125000	0.03288862	0.03289247	1.171137024611e-04
11	0.34375000	0.04017137	0.04017580	1.102745239021e-04
13	0.40625000	0.04743863	0.04744348	1.022833220901e-04
15	0.46875000	0.05468760	0.05469270	9.325851330626e-05
17	0.53125000	0.06191549	0.06192065	8.333883609117e-05
19	0.59375000	0.06911950	0.06912453	7.268362452537e-05
21	0.65625000	0.07629687	0.07630156	6.147312282756e-05
23	0.71875000	0.08344483	0.08344900	4.990884147190e-05
25	0.78125000	0.09056062	0.09056408	3.821395634789e-05
27	0.84375000	0.09764149	0.09764409	2.663375179110e-05
29	0.90625000	0.10468472	0.10468634	1.543610844166e-05
31	0.96875000	0.11168760	0.11168815	4.912037629605e-06

Maximum Relative Error: 1.312811991005e-04

5.4 Conclusion

The Haar wavelet method is computationally efficient and the algorithm can easily be implemented on computer.

1. Numerical comparisons demonstrate that classical numerical method is reliable and accurate. Refer to examples (5.2, 5.3, 5.5, 5.6, 5.7)

2. The distinctive feature is that it can be applied, to initial and boundary value problems without transformation of BVPs into IVPs as needed for the Runge-Kutta methods. In Haar wavelet method the initial and boundary conditions intervention directly without complicated .

3. Simple applicability and fast convergence of the Haar wavelets provides a solid foundation, for using these functions in the context of numerical

approximation ordinary differential equations and partial differential equations.

4. The main advantage of this method is its simplicity and small computation costs, it is due to the sparsity of the transform matrices and to the small number of significant wavelet coefficients, Hence, the present method is a very reliable, simple, fast, minimal computation costs and flexible.

5. Our computationally result shows that, the convergence the method is quadratic the infinite norm but the need investigation.

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جامعة النجاح الوطنية

كليه الدراسات العليا

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2015

ب

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إعداد

يوسف مصطفى يوسف احمد بشارات

إشراف

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الملخص

في هذه الأطروحة، استخدمت طريقة عددية جديدة نسبياً في حل المعادلات التفاضلية من النوع الخطي وهي طريقة مويجات هار. وتم عمل مقارنه عدديه بين الطريقة الجديدة وبعض الطرق العددية التقليدية المعروفة لحل المعادلات التفاضلية الخطية. وكان الهدف اثبات فعالية الطريقة المستخدمة وتميزها على الطرق التقليدية الاخرى. وتبين ان الطريقة الجديدة ابسط وتعطي نتائج قريبه او افضل من الطرق التقليدية وذلك حسب المعادلة التفاضلية.