An-Najah National University
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On Fuzzy Metric Spaces and their Applications in Fuzzy Environment

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الإقرار

أنا الموقعة أدناه، مقدمة الرسالة التي تحمل العنوان:

On Fuzzy Metric Spaces and their Applications in Fuzzy Environment

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On Fuzzy Metric Spaces and their Applications in Fuzzy Environment

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Abstract

In this thesis, the fuzzy metric spaces were investigated using different definitions and point of views. Some were applied on regular sets while others were applied on sets of fuzzy points.

The concept of complement of fuzzy metric spaces using fuzzy scalars were studied and parallel results in classical analysis were found under fuzzy Setting .

And finally , Fuzzy fixed point theorems on fuzzy metric space were proved .

Introduction

The concept of fuzzy sets was first introduced by L.Zadah in 1965 in his famous paper [26] . In this thesis the concept of fuzzy metric spaces was investigated .

In chapter one, we concentrate on the basic definitions of fuzzy sets and related concepts including fuzzy points, fuzzy functions and their properties .

In chapter two, we went over a special type of fuzzy sets, namely , fuzzy numbers and using them to define a fuzzy metric on a set of fuzzy points .

In chapter three , a fuzzy scalar is defined and has been used to measure the distance between two fuzzy points in the sense of Xia and Guo(2003) [12] . This concept goes along and parallel the distance in classical metric spaces .Also, the concept of Cauchy sequence and completeness of fuzzy metric spaces was investigated as well as the fuzzy topology induced by a fuzzy metric space.

Finally in chapter four , A new definition of fuzzy metric is presented which is an extension of classical metric on a set X , with applications in fixed point theory and other concepts in classical analysis.

Chapter one

Fuzzy Sets and Fuzzy Functions

Chapter one

Fuzzy Sets and Fuzzy Functions

Introduction

Fuzzy sets, in Mathematics, are sets having elements with a membership degree. This concept of sets was first generalized by professor Lotfi A.Zaden in 1965 in his famous paper [26] , where the concept of fuzzy sets was introduced, it was specifically designed for representing uncertainty in mathematics and for dealing with vagueness in many real life problems, it is suitable for approximating reasoning mathematical models that are hard to derive or giving a decision with incomplete information. In classical set theory, an element either belongs or doesn’t belong to the set, it is not the case in fuzzy setting, here, it has membership degree between zero and one, which describes the new definition of the Characteristic function in this chapter we will first give definitions of fuzzy sets, and then we show some operations on them and properties involving these operations.

Also we will introduce the concept of fuzzy points as especial case of fuzzy subsets, and then we define fuzzy function as an extension of functions between pairs of sets and explore the properties of fuzzy operations of fuzzy sets and fuzzy points on fuzzy functions.

1.1 fuzzy sets and fuzzy operations

In set theory a subset A of a set X can be identified with the Characteristic function $\mathbf{X_A}$ that maps X to {0,1} in away where all elements of A go to 1, while X-A elements go to 0 .

i.e. $$X_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \in X\text{-}A \end{cases}$$

And there for, there is a natural 1-1 correspondence between the family of all subsets of X and the family of the characteristic functions on X.

Definition 1.1.1

Let X be a set , a fuzzy subset of X is a function A that Maps X to the closed interval [0, 1] .In other words, A: X $\rightarrow$[0, 1], and A(x) is called the grade of membership of the element x .

Example 1.1.2

Example of fuzzy subsets of X = {a, b, c}

A: $a \rightarrow .1$
$b \rightarrow .4$
$c \rightarrow .2$

B: $a \rightarrow .8$
$b \nearrow$
$c \rightarrow 1$

C: $a \rightarrow 1$
$b \nearrow$
$c \rightarrow 0$

Definition 1.1.3

Regular subsets of X are a special case of fuzzy sets called crisp fuzzy sets where $\mathbf{A}(x) \in \{0,1\}$

Example 1.1.4

Let X= {a, b, c}

The regular subset A of X can be

$$\mathbf{A}:\ \begin{array}{lll} a & \longrightarrow & 1 \\ b & \nearrow & \\ c & \longrightarrow & 0 \end{array}$$

In this example:

$a \in A \equiv A(a) = 1$

$c \notin A \equiv A(c) = 0$

We use different ways to represent a fuzzy sebset of X, in the following example we describe some of this ways:

Example 1.1.5

Consider the regular set X where X= {a, b, c, d, e} and let A be the fuzzy subset of X that maps X to [0, 1] by mapping:

$a \longrightarrow 0.1$, $b \longrightarrow 0.8$, $c \longrightarrow 0.5$, $d \longrightarrow 0$, and $e \longrightarrow 0.4$.

We may represent A as the set of ordered pairs:

$\mathbf{A}= \{(a, 0.1), (b, 0.8), (c, 0.5), (d, 0), (e, 0.4)\}$

Using regular set notation or we may write it as

$A = \{a_{0.1}, b_{0.8}, c_{0.5}, d_0, e_{0.4}\}$

1.2 operations on fuzzy sets

After these new concepts of fuzzy sets were defined, suitable operations on them should be performed that extend the usual operations on sets including the union, intersection, and complementation as follows:

Definition 1.2.1 [22]

Let A and B be two fuzzy subsets of X. $A \cap B$, $A \cup B$,

And A^C are fuzzy subsets of X defined as follows:

$(A \cap B)(x) = \min\{A(x), B(x)\}$

$(A \cup B)(x) = \max\{A(x), B(x)\}$

$A^C(x) = 1 - A(x)$

These definitions are generalized to any number of fuzzy subsets of X, so; for any family $\{A_\sim : \sim \in \Delta\}$ of fuzzy subsets of X, where Δ is an indexing set, we define:

$$\Big(\bigcup_{\sim} A_{\sim}\Big)(x) = \mathbf{SUP}\,\{A_\sim(x) : \sim \in \Delta\}$$

$$\Big(\bigcap_{\sim} A_{\sim}\Big)(x) = \mathbf{inf}\,\{A_\sim(x) : \sim \in \Delta\}$$

We illustrate the previous definitions by the following examples:

Example 1.2.2

Take the fuzzy subsets

A= {(a, 0.3), (b, 0.8), (c, 0), (d, 0.98)}

B= {(a, 0.8), (b, 0.1), (c, 0.1), (d, 0.3)}

Then: $A \cap B$ = {(a, 0.3), (b, 0.1), (c, 0), (d, 0.3)}

$A \cup B$ = {(a, 0.8), (b, 0.8), (c, 0.1), (d, 0.98)}

Example 1.2.3

Take an infinite number of fuzzy subsets.

Let X= {a, b}

A1 = {(a, 0.49), (b, 0.21}

A2 = {(a, 0.499), (b, 0.201)}

A3 = {(a, 0.4999), (b, 0.2001)

$\vdots$

Then $\bigcup_{i=1}^{\infty} A_i$ = {(a, 0.5), (b,0.21)}

And $\bigcap_{i=1}^{\infty} A_i$ = {(a, 0.49), (b, 0.2)}

Let A and B be two fuzzy subsets of X, we have:

1. $(A \cap B)^C (x) = (A^C \cup B^C) (x)$.

2. $(A \cup B)^{C}(x) = (A^{C} \cap B^{C})(x)$.

Proof:

1) $(A \cap B)^{C}(x) = 1 - \min\{A(x), B(x)\}$

$$= \begin{cases} 1 - A(x) & \text{if } A(x) \leq B(x) \\ 1 - B(x) & \text{if } B(x) \leq A(x) \end{cases}$$

$$= \begin{cases} 1 - A(x) & \text{if } 1 - A(x) > 1 - B(x) \\ 1 - B(x) & \text{if } 1 - B(x) > 1 - A(x) \end{cases}$$

$= \max\{1 - A(x), 1 - B(x)\}$

$= \max\{A^{C}(x), B^{C}(x)\}$

$= (A^{C} \cup B^{C})(x)$

2) $\{(A \cup B)^{C}(x) = 1 - \max\{A(x), B(x)\}$

$$= \begin{cases} 1 - A(x) & \text{if } A(x) \geq B(x) \\ 1 - B(x) & \text{if } B(x) \geq A(x) \end{cases}$$

$$= \begin{cases} 1 - A(x) & \text{if } 1 - A(x) < 1 - B(x) \\ 1 - B(x) & \text{if } 1 - B(x) < 1 - A(x) \end{cases}$$

$= \min\{1 - A(x), 1 - B(x)\}$

$= \min\{A^{C}(x), B^{C}(x)\}$

$= (A^{C} \cap B^{C})(x)$

This theorem can be generalized to any family of fuzzy subsets of X. Specifically :

$(\bigcup \tilde{A})^C = (\bigcap \tilde{A}^C)$ and $(\bigcap \tilde{A})^C = (\bigcup \tilde{A}^C)$.

Now we compare two fuzzy subsets of a set X as one of them Containing the other as follows:

Definition 1.2.4 [22]

Let A , B be two fuzzy subsets of X, we say $A \subseteq B$

to mean $A(x) \leq B(x)$ for all $x \in X$.

For example

Consider **X** = {a,b,c,d} , and let A ={(a,0.1), (b,0.8), (c,0),(d,0)}

B={ (a,0.4), (b,0.8), (c,0.1), (d,0)}. then clearly $A \subseteq B$

Definition 1.2.5 [22]

The α-level of A denoted by A^α is a subset of X, where the grade of membership of its elements ≥ α . That is,

$A^\alpha = \{x \in X: A(x) \geq \alpha\}$, where $\alpha > 0$

We define the zero level of A by :

A^{zero} = the closure of $\{x \in X: A(x)>0\})$ in R

The support of A is defined as the set of all elements of X with non zero membership and denoted by supp of A that is,

Supp (A) =$\{\overline{x \in X : A(X) > 0}\}$

The following example displays some α –levels of some fuzzy subsets:

Let A = { (a,0.4), (b,0.7), (c,0.3), (d,0.2) } be a fuzzy subset

Of X = {a,b,c,d} then the 0.3- level = $A^{.3}$ = {a,b,c}

The 0 .1- level = $A^{.1}$ = {a,b,c,d}. and the support is :

A= X= {a,b,c,d}.

1.3 fuzzy points and fuzzy singletons

Definition 1.3.1 [23]

Let X be a set and let a $\in$X,

we define the fuzzy point (a , λ) as a fuzzy subset **p**

Of X such that **p(a)** = λ , p(X - {a}) = 0, where

$0<\lambda<1$.

A fuzzy singleton is **q** = (a , λ) but $0< \lambda \leq 1$

Definition 1.3.2 [23]

Let **p** = (a , λ) be a fuzzy point

P$\in$ A if and only if $\lambda \leq A(a)$

For example:

X= { a,b,c }

A= {(a ,0.4) , (b,0.8), (c,0.1)}

Then (a, 0.3) $\in$ A but (a , 0.5) $\notin$ A .

Definition 1.3.3 [16]

A fuzzy singleton (x , r) in X is said to be quasi- coincident with a fuzzy set **M** in X if and only if r +M(x) >1 and this is denoted by

(x , r) **Q** M **.**

Remark : it is clear that x_a quasi M $\Longleftrightarrow$ $x_a \notin M^C$

Note : (Q≡ quasi).

Definition 1.3.4 [16]

A fuzzy subset **A** in X is called quasi- coincident with a fuzzy subset **B** in X (denoted by A quasi B) if and only if

A (x)+ B(x) > 1 for some x in X.

1.4 fuzzy functions

Now, we introduce the fuzzy function concept between two families of fuzzy subsets corresponding to a function between two crisp sets.

Definition 1.4.1 [22]

h : X $\longrightarrow$ Y be any function.

for any fuzzy subset A of X , we define :

h*: H(X) $\longrightarrow$ H(Y) , by h*(A) to be the fuzzy subset of Y

defined by : $h^*(A)(y) = \begin{cases} \sup\{A(x) : x \in h^{-1}(y)\} & \text{if } h^{-1}(y) \neq \Phi \\ 0 & \text{if } h^{-1}(y) = \Phi \end{cases}$

and we define the fuzzy function $(h^*)^{-1}$ as$(h^*)^{-1}$(B) for any fuzzy subset B of Y by:

$(h^*)^{-1}(B)(x) = B(h(x))$

Now, we consider examples that clarify the above definition.

Example 1.4.2 [1]

Take X= {a, b, c, d}. y = [u, v, m]

And h: X $\longrightarrow$ Y by : a $\longrightarrow$ u, b $\longrightarrow$ v, c $\longrightarrow$ v and d $\longrightarrow$ v.

Let A be the fuzzy subset of X such that :

A= {(a, 0.2), (b, 0.5), (c, 0.6), (d, 0)}, then h*(A) is the fuzzy subset

of Y defined as: h*(A): Y $\longrightarrow$ [0, 1]

u $\longrightarrow$ 0.2, v $\longrightarrow$ max{0.5, 0.6,0} = 0.6, and m $\longrightarrow$ 0

Example 1.4.3 [2]

Let X= {a, b, c, d}, Y= {u, v, m} and h: X $\longrightarrow$ Y.

be the function that maps a to u and b, c and d to v, and let

B: Y$\longrightarrow$ [0, 1] to be the fuzzy subset of Y that maps u to 0.3, v to 0.5,

and m to 0.8

Then $(h^*)^{-1}(B)$: X $\longrightarrow$ [0, 1]

a $\longrightarrow$ 0.3, b$\longrightarrow$ 0.5, c $\longrightarrow$ 0.5 and d $\longrightarrow$ 0.5

Theorem1.4.4 [22]

(1) If p = (a , λ) is a fuzzy point in x, with support a, and with

Value = λ , then h*(p) is a fuzzy point in Y, call it q, where

$h^*(p) = h(a)_a = q$, such that h(a) is the support of q , and λ is

the value of q .

Proof:

if $h^{-1}(y) = \Phi$, q(y) =0

if $h^{-1}(y) \neq \Phi$, $q(y) = \sup \{p(x) : x \in h^{-1}(y)\}$ here, there are two cases :

● case one : if $a \in h^{-1}(y)$

$q(y) = \sup \{p(x) : x \in h^{-1}(h(a))\} = \{\lambda, 0, 0, ...\} = \lambda$

● case two a $\notin h^{-1}(y)$

$q(y) = \sup = \{0,0,...\} = 0$

(2) if q= (b , r) fuzzy point in Y then $h^{-1}(q)$ may not be fuzzy point in X.

The following examples explain this result.

Example1.4.5

Suppose $h^{-1}(b)$ is not a singleton, say $h^{-1}(b) = \{\alpha, ß\}$

then $h^{*-1}(q) = \{\alpha_r, ß_r, 0, 0, \ldots\}$ which is not a fuzzy point .

Example 1.4.6

IF $f^{-1}(b) = \Phi$, then $f^{*-1}(q) = \Phi$ which is not a fuzzy point According to the previous example if $f^{-1}(b)$; where $q = b_r$; is a singleton then $f^{*-1}(q)$ is a fuzzy point in X.

The following theorem shows the effect of fuzzy functions on the quasi- coincident relation between a fuzzy point and a fuzzy set .

Theorem 1.4.7 [25]

Let $h : X \longrightarrow Y$ be a function, then for any fuzzy point $p = (a , \lambda)$

and for any fuzzy subset A of X, we have:

if p quasi A then $h^*(p)$ quasi $h^*(A)$.

Proof

Let $p=(a, \lambda)$ and $h(p) = h(a)$ a

Since p quasi A then : $\lambda + A(a) > 1$

Consider $\lambda + h(A)(h(a))$

a $+ h(A)(h(a)) = \lambda + \sup \{A(x) : x \in h^{-1}(h(a))\}$

$\geq \lambda + A(a) > 1$

Theorem 1.4.8

If $q=b_{\lambda}$ and $f^{-1}(b)$ is a singleton ($f^{-1}(b) = \{a\}$) then $f^{*-1}(q)$ is the fuzzy point = a a and in this case : if q quasi B then $f^{*-1}(q)$ quasi $f^{*-1}(B)$

Proof

We have q quasi B which means $\lambda + B(b) > 1$

Now ,

$\lambda + f^{*-1}(b)(a) = \lambda + B(f(a))$

$= \lambda + B(b) > 1$

That is , $f^{*-1}(q)$ quasi $f^{*-1}(B)$.

Theorem 1.4.9

Let $h: X \longrightarrow Y$ be a function

A , B fuzzy subsets of X then :

1) $h^*(A \cup B) = h^*(A) \cup h^*(B)$

2) $h^*(A \cap B) \subseteq h^*(A) \cap h^*(B)$

Proof:

1) $h^*(A \cup B)(y)$

If $h^{-1}(y) = \Phi$ then $h^*(A \cup B)(y) = 0$

Now, $(h^*(A) \cup h^*(B)(y) = \max \{h^*(A)(y), h^*(B)(y)\}$

$= \max \{0 , 0\}$

$= 0$

If $h^{-1}(y) \neq \emptyset$ and let $h^{-1}(y) = \{x_1, x_2, x_3\}$

$h^*(A \cup B)(y) = \max\{ (A \cup B)(x) : x \in h^{-1}(y) \}$

$= \max \{\max\{A(x), B(x) : x \in h^{-1}(y)\}$

$= \max \{a_1 v b_1, a_2 v b_2, a_3 v b_3, \ldots\}$

Where, $A(x_1) = a_1$, $B(x_1) = b_1$, $A(x_2) = a_2$, $B(x_2) = b_2$, $A(x_3) = a_3$, $B(x_3) = b_3$

$= \max\{a_1 v a_2 v a_3 v \ldots, b_1 v b_2 v b_3 v \ldots\}$

$= \max \{\max\{A(x_1) , A(x_2), \ldots , \max\{B(x_1), B(x_2), \ldots\}$

$= \max \{h^*(A)(y), h^*(B)(y)\}$

$= (h^*(A) \cup h^*(B))(y)$

$\therefore\ h^*(A\cup B) = h^*(A) \cup h^*(B)$.

Proof

2) want to prove $h^*(A\cap B) \subseteq h^*(A) \cap h^*(B)$

If $h^{-1}(y) = \Phi$ then $h^*(A\cap B)(y) =0$

Also, $(h^*(A) \cap h^*(B)(y) = \min \{h^*(A)(y), h^*(B)(y)\}$

$=\min \{0,0\}$

$=0$

i.e $h^*(A\cap B) = h^*(A) \cap h^*(B)$

If $h^{-1} (y) \neq ø$

$h^*(A\cap B)(y)= \max\{ (A\cap B)(x) :x\in h^{-1}(y)\}$

$= \max\{ \min A(x) ,B(x) : x\in h^{-1}(y)\}$

$=\max\{a_1\Lambda b_1, a_2\Lambda b_2,a_3\Lambda b_3\}$

Now, $\max \{a_1\Lambda b_1, a_2\Lambda b_2, a_3\Lambda b_3\} \leq \min \{a_1 v a_2 v a_3 , b_1 v b_2 v b_3 \}$

Because, $a_1\Lambda b_1 \leq a_1 v\, a_2 v\, a_3$ | $a_1 \Lambda b_1 \leq b_1 v\, b_2 v\, b_3$

$a_2 \Lambda b_2 \leq a_1 v\, a_2 v\, a_3$ | $a_2 \Lambda b_2 \leq b_1 v\, b_2 v\, b_3$

$a_3\Lambda b_3\leq a_1 v\, a_2 v\, a_3$ | $a_3 \Lambda b_3 \leq b_1 v\, b_2 v\, b_3$

i.e $h^*((A\cap B)(y) \leq \min\{h^*(A)(y), h^*(B)(y)\}$

$$= (h^*(A) \cap h^*(B))(y)$$

i.e. $h^*(A\cap B) \subseteq h^*(A) \cap h^*(B)$

Remak:

Example where $h^*(A\cap B) \neq h^*(A) \cap h^*(B)$

$X = \{a, b\}$, $y = \{c\}$

$\mathbf{W}$: $x \longrightarrow y$

$$\begin{tikzcd} a \arrow[r] & c \\ b \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} .4 & a \arrow[l, "A"'] \arrow[dr] & \\ & & c \\ .3 & b \arrow[l] \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} .2 & a \arrow[l, "B"'] \arrow[dr] & \\ & & c \\ .4 & b \arrow[l] \arrow[ur] & \end{tikzcd}$$

now, $h^*(A) : c \longrightarrow .4$

$h^*(B) : c \longrightarrow .4$

$h^*(A) \cap h^*(B): c \longrightarrow .4$

also

$$\begin{tikzcd} .2 & a \arrow[l, "A\cap B"'] \arrow[dr] & \\ & & c \\ .3 & b \arrow[l] \arrow[ur] & \end{tikzcd}$$

$h^*(A\cap B): c \longrightarrow .3$

i.e $h^*(A\cap B) \neq h^*(A) \cap h^*(B)$.

Theorem 1.4.10

let $h : X \longrightarrow Y$ be a function

A , B fuzzy subsets of X and

L , M be fuzzy subsets of Y . Then :

1. $(h^*)^{-1} (L \cup M) = (h^*)^{-1} (L) \cup (h^*)^{-1}(M)$
2. $(h^*)^{-1}(L \cap M)=(h^*)^{-1}(L) \cap (h^*)^{-1} (M)$

Proof :

(1) $(h^*)^{-1} (L \cup M) \Rightarrow (h^*)^{-1} (L \cup M)(x) = (L \cup M)(h(x))$

$\Rightarrow (L \cup M)(h(x)) =$

$= \max \{L(h(x)), M(h(x)\}$

$= \max \{(h^*)^{-1} (L)(x), (h^*)^{-1} (M)(x)\}$

$= ((h^*)^{-1}(L) \cup (h^*)^{-1}(M)) (x)$

$= (h^*)^{-1}(L) \cup (h^*)^{-1}(M)$

i.e $(h^*)^{-1}(L \cup M)=(h^*)^{-1}(L) \cup (h^*)^{-1}(M)$

(2) $(h^*)^{-1}(L \cap M)=(h^*)^{-1} (L \cap M)(X)= (L \cap M) h(x)$

$= \min \{L(h(x)), M(h(x))\}$

$= \min ((h^*)^{-1}(L)(X),(h^*)^{-1}(M)(x)\}$

$= ((h^*)^{-1}(L) \cap (h^*)^{-1}(M)(x)$

$= (h^*)^{-1}(L) \cap (h^*)^{-1}(M)$

Chapter Two

Fuzzy Number and Fuzzy Metric

Chapter Two

Fuzzy Number and Fuzzy Metric

Introduction

How to define fuzzy metric is one of the fundamental problems in fuzzy mathematics which is wildly used in fuzzy optimization and pattern recognition . there are two approaches in this field till now.

One is using fuzzy numbers define metric in ordinary spaces . firstly proposed by Kaleva (1984) [13] .

Fuzzy topology induced by fuzzy metric spaces ,fixed point theorems and other properties of fuzzy metric spaces are studied by a few researchers , Felbin (1992) [3] , George (1994) [4] , Hadzic (2002)[11] .

The other one is using real numbers to measure the distances between fuzzy sets .The references of this approach can be referred to , for instance, Dia (1990) , Boxer (1997) , Fan (1998) , Brass (2002) , results of these researches have been applied to many practical problems in fuzzy environment, while usually different measures are used in different problems in other words, there does not exist a uniform measure that can be used in all kinds of fuzzy environments.

Therefore, it is still interesting to find some kind of new fuzzy measure such that it may be useful for solving some problems in fuzzy environment.

2.1: Fuzzy Number

Definition 2.1.1

A fuzzy number A is a fuzzy subset of R, i.e. $A : R \longrightarrow [0, 1]$

Such that, there exists a, b, c, and d $\in$ R such that $a < b < c < d$

with:

$$A(X) = \begin{cases} 0 & , \text{if } x \leq a , x \geq d \\ \text{Increasing} \nearrow & , \text{if } x \in [a, b] \\ \text{Decreasing} \searrow & , \text{if } x \in [c, d] \\ 1 & , \text{if } x \in [b, c] \end{cases}$$

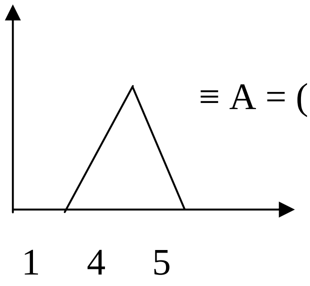
Note : if b=c , and between a and b there are linear function also between b and c there are linear function then we say A is a triangular fuzzy number.

Example 2.1.2

Given the fuzzy number:

$$A(x) = \begin{cases} \frac{x-1}{3} & ,\text{if } \in x \ [1,4] \\ 5 - x & , \text{if } x \in [4, 5] \\ 0 & ,\text{other wise} \end{cases}$$

s.t :



$\equiv A = (1, 4, 5)$

In the previous example support of $A \equiv [1, 5] \equiv A^{\text{zero – level}}$.

Also , $A^{.2}$ = .2 – level s.t : $A^{.2} = \{ x \in X : A(x) \geq .2 \}$

$.2 = \dfrac{x-1}{3} \longrightarrow x_1 = 1.6$, $.2 = 5 - x \longrightarrow x_2 = 4.8$

$\therefore$ .2 – level $\equiv A^{.2} = [1.6, 4.8]$.

Also , we can solve another example to find $A^{.5}$ as follows :

$.5 = \dfrac{x-1}{3} \longrightarrow x_1 = 2.5$, $.5 = 5 - x \longrightarrow x_2 = 4.5$

Ingeneral, we can find $A^{\sim}$ s.t :

$\sim\ = \dfrac{x-1}{3} \longrightarrow (x_1)^{\sim} = 1+3\sim$

$\sim\ = 5 - x \longrightarrow (x_2)^{\sim} = 5 - \sim$

Therefore ,

$A^{\sim} \equiv \sim$ - level = $[\, 1+3\sim , 5 - \sim\,]$.

2.2 Addition of fuzzy number

Definition 2.2.1

Let A , B be two fuzzy numbers , we define A+B by :

$(A+B)^{\sim} = [\, a_1^{\sim} + b_1^{\sim} \;, a_2^{\sim} + b_2^{\sim} \,]$, where $A^{\sim} = [\, a_1^{\sim} , a_2^{\sim} \,]$

$B^{\sim} = [\, b_1^{\sim} , b_2^{\sim} \,]$

Example 2.2.2

Let A , B be two fuzzy numbers defined by :

$$A(x) = \begin{cases} \frac{X-1}{3} & \text{if } x \in [1, 4] \\ 5 - x & \text{if } x \in [4, 5] \\ 0 & \text{otherewise} \end{cases}$$

$$B(x) = \begin{cases} x - 2 & \text{if } x \in [2, 3] \\ \frac{7-x}{4} & \text{If } x \in [3, 7] \\ 0 & \text{, otherewise} \end{cases}$$

We need to find $(A+B)^{\sim}$. To find $(A+B)^{\sim}$ we need firstly to find $A^{\sim}$ And $B^{\sim}$.

To find $A^{\sim}$ we let $\sim \; = \frac{X-1}{3} \longrightarrow x = 1 + 3\alpha \longrightarrow a_1^{\sim} = 1+3^{\sim}$

Also, $\alpha = 5 - x \longrightarrow x = 5 - \alpha \longrightarrow a_2^{\sim} = 5 - \alpha$

So, $A^{\sim} = [1+3\alpha, 5 - \alpha]$.

To find $B^{\sim}$ we let $\alpha = x - 2 \longrightarrow x = 2+\alpha \longrightarrow b_1^{\sim} = 2+\alpha$.

Also, $\alpha = \frac{7-x}{4} \longrightarrow x = 7 - 4\alpha \longrightarrow b_2^{\sim} = 7 - 4\alpha$.

So, $B^{\sim} = [2+\alpha, 7 - 4\alpha]$

Now ,

Let $C^{\sim} = (A+B)^{\sim}$, so C = A+B.

$C^{\sim} = [\, a_1^{\sim} + b_1^{\sim} , a_2^{\sim} + b_2^{\sim} \,] = [\, 1+ 3\alpha+2+\alpha , 5 - \alpha +7 - 4\alpha \,]$

So $C^{\sim} = [\, 3+4\alpha , 12 - 5\alpha \,]$.

In the previous example we can find (A + B)(x) = C(x) as follows :

In the last example $C^{\sim} = (A+B)^{\sim} = [\, 3+4\alpha , 12 - 5\alpha \,]$.

To find C(x) we need to use $c_1^{\sim}$, $c_2^{\sim}$.

1) $c_1^{\sim} = 3+4\alpha$, $\alpha \in [\, 0 , 1 \,]$

When $\alpha = 0$ then $c_1^0 = 3+0 = 3$.

When $\alpha = 1$ then $c_1^1 = 3 + 4 (1) = 7$.

So , when $\alpha = 0$ then $x_1 = 3$, when $\alpha = 1$ then $x_2 = 7$.

So, $x \in [3, 7]$, also we put X instead of $c_1^{\sim}$ and we put y instead Of α , then ($c_1^{\sim} = 3 + 4\alpha$) become $x = 3+4y \longrightarrow Y = \dfrac{x-3}{4}$

s.t $x \in [\, 3 , 7 \,]$.

2) $c_2^{\sim} = 12 - 5\alpha$, $\alpha \in [\, 0 , 1 \,]$

When $\alpha = 0$ then $c_2^0 = 12$

When $\alpha = 1$ then $c_2^1 = 12 - 5\,(1) = 7$

So , when $\alpha = 0$ then $x_2 = 12$, when $\alpha = 1$ then $x_1 = 7$.

So $x \in [7, 12]$, also we put x instead of $\tilde{c_2}$ and we put y instead

of α then $\tilde{c_2} = 12 - 5\alpha$, $\alpha \in [0, 1]$ become :

$$x = 12 - 5y \longrightarrow y = \frac{12 - x}{5} \ , \ x \ [7, 12] .$$

$$\text{So , } C(x) = (A+B)(x) = \begin{cases} \dfrac{x-3}{4} & , \text{ if } x \in [3, 7] \\[2ex] \dfrac{12-x}{5} & , \text{ if } x \in [7, 12] \\[2ex] 0 & , \text{ otherewise} \end{cases}$$

Remark:

In special case (linear triangular fuzzy numbers) , we mean when we have two fuzzy number $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$, then

$(A+B)(x) = (a_1 + b_1 , a_2 + b_2 , a_3 + b_3)$.

This result obtain directly without using α – level , only , when we deal with triangular fuzzy numbers .We will illustrate this remark as follows :

$$A(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1} & a_1 \le x \le a_2 \\[2ex] \dfrac{a_3-x}{a_3-a_2} & a_2 \le x \le a_3 \\[2ex] 0 & \text{Otherwise} \end{cases}$$

$$B(x) = \begin{cases} \dfrac{x-b_1}{b_2-b_1} & b_1 \le x \le b_2 \\ \dfrac{b_3-x}{b_3-b_2} & b_2 \le x \le b_3 \\ 0 & \text{Otherwise} \end{cases}$$

$A^{\alpha} = [\ \alpha\ (\ a_2 - a_1\ ) + a_1\ ,\ a_3 - \alpha\ (a_3 - a_2\ )\ ]$

$B^{\alpha} = [\ \alpha\ (\ b_2 - b_1\ ) + b_1\ ,\ b_3 - \alpha\ (b_3 - b_2\ )\ ]$

$(A + B)^{\alpha} = [\ \alpha\ (a_2 - a_1 + b_2 - b_1\ ) + a_1 + b_1\ ,\ a_3 + b_3 - \alpha\ (a_3 - a_2 + b_3 - b_2)$

Which leads :

$$(A+B)(x) = \begin{cases} \dfrac{x-(a_1+b_1)}{(a_2+b_2)-(a_1+b_1)} & , \quad a_1+b_1 \le x \le a_2+b_2 \\ \dfrac{(a_3+b_3-x)}{(a_3+b_3)-(a_2+b_2)} & , \quad a_2+b_2 \le x \le a_3+b_3 \\ 0 & , \quad \text{Otherwise} \end{cases}$$

And the following example explain this remark

Example 2 . 2 .3

Given **A** = (1 , 3, 7) , **B** = (4 , 5 , 6) two linear triangular fuzzy numbers then :

(A+B) = (1+4 , 3+5 , 7+6) = (5 , 8 , 13) .

$$(A+B)(x) = \begin{cases} \dfrac{x-5}{3} & , x \in [5, 8] \\ \dfrac{13-x}{5} & , x \in [8, 13] \\ 0 & , \text{other wise} \end{cases}$$

If we solve the previous example by α – level we will obtain the same Answer as follows :

A = (1 , 3 , 7) , **B** = (4 , 5 , 6)

We can write A and B as follows :

$$A(x) = \begin{cases} \dfrac{x-1}{2} & , \text{ if } x \in [1,3] \\ \dfrac{7-x}{4} & , \text{ if } x \in [3,7] \\ 0 & , \text{ otherwise} \end{cases}$$

$$B(x) = \begin{cases} x-4 & , \text{ if } x \in [4,5] \\ 6-x & , \text{ if } x \in [5,6] \\ 0 & , \text{ otherwise} \end{cases}$$

Firstly , we need to find $(A+B)^{\sim} \equiv C^{\sim} = [\, a_1^{\sim} + b_1^{\sim} , a_2^{\sim} + b_2^{\sim} \,]$.

s.t : $A^{\sim} = [\, a_1^{\sim} , a_2^{\sim} \,]$, $B^{\sim} = [\, b_1^{\sim} , b_2^{\sim} \,]$

to find $A^{\sim}$ we let $\alpha = \dfrac{x-1}{2} \longrightarrow x = 1+2\alpha \longrightarrow a_1^{\sim} = 1+2\alpha$

Also , $\alpha = \dfrac{7-x}{4} \longrightarrow x = 7-4\alpha \longrightarrow a_2^{\sim} = 7-4\alpha$

$A^{\sim} = [\, 1+2\alpha , 7-4\alpha \,]$.

To find $B^{\sim}$ we let $\alpha = x-4 \longrightarrow x = \alpha+4 \longrightarrow b_1^{\sim} = \alpha+4$

Also , $\alpha = 6-x \longrightarrow x = 6-\alpha \longrightarrow b_2^{\sim} = 6-\alpha$.

$\tilde{B} = [\alpha + 4, 6 - \alpha]$

$\tilde{C} = [1+2\alpha+\alpha+4, 7 - 4\alpha + 6 - \alpha]$

$= [\, 5+3\alpha \, , 13 - 5\alpha \,]$

To find (A+B)(x) = C(x) we use $\tilde{c_1}$ and $\tilde{c_2}$.

$\tilde{c_1} = 5 + 3\alpha$, $\alpha \in [\,0 ,1\,]$, when $\alpha = 0$ then $c_1^0 = 5 \longrightarrow x_1 = 5$.

when $\alpha = 1$ then $c_1^1 = 5+3(1) = 8 \longrightarrow x_2 = 8$.

replace $\tilde{c_1}$ by x and α by y then $\tilde{c_1} = 5+3\alpha$ become :

$x = 5+3y \longrightarrow y = \dfrac{x-5}{3}$, $x \in [\,5 , 8\,]$.

$\tilde{c_2} = 13 - 5\alpha$, $\alpha \in [\,0 , 1\,]$, when $\alpha = 0$ then $c_2^0 = 13 \longrightarrow x_2 = 13$

when $\alpha = 1$ then $c_2^1 = 13 - 5(1) = 8 \longrightarrow x_1 = 8$.

so $\tilde{c_2} = 13 - 5\alpha$ become $x = 13 - 5y \longrightarrow y = \dfrac{13-x}{5}$, $x \in [\,8 , 13\,]$

$$\Longrightarrow C(x) = (A+B)(x) = \begin{cases} \dfrac{x-5}{3} , & x \in [\,5 , 8\,] \\ \dfrac{13-x}{5} , & x \in [\,8 , 13\,] \\ 0 & , \text{otherwise} \end{cases}$$

There fore , the two answers are the same .

2.3 Fuzzy Metric

Definition 2.3.1

Let X_λ , y_r be two fuzzy points , let $k \in \left(0, \frac{1}{2}\right]$ be a fixed number $\lambda > 0$, $r > 0$ .we define the k – distance by :

$$d_k(x_\lambda, y_r) = \begin{cases} (\lambda \wedge r\ ,\ k(\lambda \wedge r)+(1-k)(\lambda \vee r)\ ,\ \lambda \vee r\ ) & , \text{if } x \neq y \\ (0\ ,\ (1-k)\ |\lambda - r|\ ,\ |\lambda - r|\ ) & , \text{if } x = y \end{cases}$$

$d_k(x_\lambda, y_r)$ is afuzzy metric such that :

1) $d_k(x_\lambda, y_r)$ is a fuzzy number A , where support of $A \geq 0$

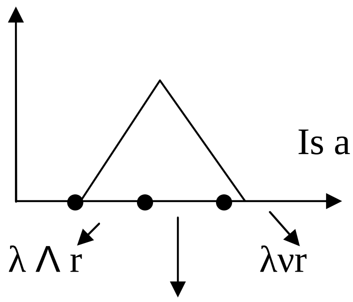
2) $d_k(x_\lambda, y_r) = \bar{0} = (0, 0, 0)$ if and only if $x_\lambda = y_r$.

3) $d_k(x_\lambda, y_r) = d_k(y_r, x_\lambda)$.

4) $d_k(x_\lambda, z_t) \leq d_k(x_\lambda, y_r) + d_k(y_r, z_t)$.

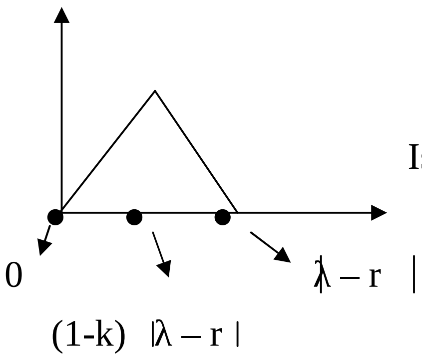
We can show that the previous conditions are satisfied as follows :

1) $d_k(x_\lambda, y_r)$ is afuzzy number



Is a fuzzy number from definition 2.1.1

$k(\lambda \wedge r)+(1-k)(\lambda \vee r)$



Is a fuzzy number from definition 2.1.1

2) *a) if $d_k(x_\lambda, y_r) = \bar{0} = (0, 0, 0)$ then $(x_\lambda = y_r)$

Proof :

We have two cases :

Case 1 : x≠y

$(\lambda \wedge r, k(\lambda \wedge r)+(1-k)(\lambda \vee r), \lambda \vee r) = (0, 0, 0)$

i.e $\lambda = 0$ and $r = 0$, which is contradiction because $\lambda > 0$, $r > 0$.

therefore x≠y is not true and that is mean the two fuzzy point are

not distincit , they are the same . i.e $(x_\lambda = y_r)$.

i.e $\lambda \wedge r$, $\lambda \vee r$, both not equal zero we could not apply rule 1 of the definition of $d_k(x_\lambda, y_r)$.

Case 2 : x=y

$d_k(x_\lambda, y_r) = (0, 0, 0) = (0, (1-k)|\lambda - r|, |\lambda - r|)$

so $(1-k)|\lambda - r| = 0$, but $(1-k) \neq 0$

$\Longrightarrow \lambda - r = 0$

$\lambda = r$, but $x = y$

$x_\lambda = y_r$.

*b) if $x_\lambda = y_r$ then $d_k(x_\lambda, y_r) = (0, 0, 0)$

Now , $x_\lambda = y_r \Longrightarrow x = y$ and $\lambda = r$.

i.e $d_k(x_\lambda, y_r) = (0, (1-k)|\lambda - r|, |\lambda - r|) = (0, 0, 0)$.

therefore , $d_k(x_\lambda, y_r) = \overline{0} = (0, 0, 0)$ if and only if $x_\lambda = y_r$.

3) *c) $d_k(x_\lambda, y_r) = \begin{cases} (\lambda \wedge r, k(\lambda \wedge r)+(1-k)(\lambda \vee r), \lambda \vee r), & \text{if } x \neq y \\ (0, (1-k)|\lambda - r|, |\lambda - r|), & \text{if } x = y \end{cases}$

*d) $d_k(y_r, x_\lambda) = \begin{cases} (r \wedge \lambda, k(r \wedge \lambda)+(1-k)(r \vee \lambda), r \vee \lambda), & \text{if } x \neq y \\ (0, (1-k)|r - \lambda|, |r - \lambda|), & \text{if } x = y \end{cases}$

When $x \neq y$ in c and d :

$\lambda \wedge r = r \wedge \lambda$

$k(\lambda \wedge r)+(1-k)(\lambda \vee r) = k(r \wedge \lambda)+(1-k)(r \vee \lambda)$,

$\lambda \vee r = r \vee \lambda$.

also , when $x = y$ in c and d :

$0 = 0$, $(1-k)|\lambda - r| = (1-k)|r - \lambda|$, $|\lambda - r| = |r - \lambda|$.

So , $d_k(x_\lambda, y_r) = d_k(y_r, x_\lambda)$.

4) to prove 4 we have consider five cases , where every case contains six cases as follow :

case 1 : x , y , z are distinct

case 2: $x=y\neq z$

case 3: $x=z\neq y$

case 4: $y=z\neq x$

case 5: $x=y=z$

Now , in every case of the previous cases , we consider the following six cases :

$\underline{\lambda<r<t}$, $\underline{\lambda<t<r}$, $\underline{r<\lambda<t}$, $\underline{r<t<\lambda}$, $\underline{t<\lambda<r}$, $\underline{t<r<\lambda}$.

in this section we will prove some of those cases (not all)

case 1 : x , y ,z are distinct with $t<\lambda<r$

let $A = d_k(x_\lambda , z_t) = \lambda \wedge t, k(\lambda \wedge t)+(1-k)(\lambda \vee t) , \lambda \vee t$

$$= (t , kt + (1- k) \lambda , \lambda)$$

$$B = d_k(x_\lambda , y_r) + d_k(y_r , z_t)$$

$$= \mathbf{(} (\lambda , k\lambda+(1-k)r , r) \mathbf{)} \mathbf{+} \mathbf{(} (t , kt +(1-k)r , r) \mathbf{)}$$

$$= (\lambda+t , k (\lambda+t) + (1 -k) 2r , 2r)$$

Now , we want to show that :

$A \leq B$.

Let

$a_1 = t$, $a_2 = kt + (1- k)\lambda$, $a_3 = \lambda$

and $b_1 = \lambda+t$, $b_2 = k(\lambda+t) + (1-k)2r$, $b_3 = 2r$

to prove that $A \leq B$ we want to prove that :

$a_1+a_2+a_3 \leq b_1+b_2+b_3$

or $b_1+b_2+b_3 - a_1-a_2-a_3 \geq 0 \equiv (b_3 - a_3) + (b_2 - a_2) + (b_1 - a_1) \geq 0$.

$\lambda + t + k(\lambda+t) + 2r(1-k) + 2r - t - kt - (1-k)\lambda - \lambda$

$\lambda + t + k\lambda + kt + 2r - 2kr + 2r - t - kt - \lambda + k\lambda - \lambda$

$4r - 2kr - \lambda + 2k\lambda$

$= 2k\lambda - \lambda + 4r - 2rk$

$= 2k\lambda - \lambda + 3r + r - 2rk$

$= (-\lambda + r) + 2k\lambda + 3r - 2rk$ s.t :

* $(-\lambda+r) > 0$ because from given $r > \lambda$.

* $2kr > 0$ because from given $r > 0$ and $k \in (0, \frac{1}{2}]$.

* $3r - 2rk > 0$ because :

$0 < k < \frac{1}{2}$

$= 0 < 2rk < r$

$= 0 > -2rk > -r$

$= 3r > 3r - 2rk > 3r - r$

$= 3r > 3r - 2rk > 2r$

$= 2r < 3r - 2rk < 3r$, since $r > 0$. so $2r > 0$, $3r > 0$.

Therefore $3r - 2rk > 0$.

Case 5: $x=y=z$ with $\lambda < t < r$

let $A = d_k(x_\lambda, z_t) = (0, (1-k)|\lambda - t|, t - \lambda)$,since $t > \lambda$:

$(0, (1-k)|\lambda - t|, t - \lambda)$ become $(0, (1-k)(t-\lambda), t - \lambda)$

$B = d_k(x_\lambda, y_r) + d_k(y_r, z_t)$

$= \big((0, (1-k)(r-\lambda), r-\lambda) \big) + \big((0, (1-k)(r-t), r-t) \big)$

$= (0, (1-k)(2r-\lambda-t), 2r-\lambda-t)$

Let : $b_1 = 0$, $b_2 = (1-k)(2r-\lambda-t)$, $b_3 = 2r-\lambda-t$

$a_1 = 0$, $a_2 = (1-k)(t-\lambda)$, $a_3 = t - \lambda$

$b_1 + b_2 + b_3 = 0 + (1-k)(2r-\lambda-t) + 2r-\lambda-t$

$\geq 0 + (1-k)(2t-\lambda-t) + 2t-\lambda-t$

$= 0+(1-k)(t-\lambda)+(t-\lambda) = a_1+a_2+a_3$

SO , $A \leq B$.

Chapter Three

Fuzzy Scalar and Fuzzy Metric

Chapter Three

Fuzzy Scalar and Fuzzy Metric

Introduction

In this chapter we will use fuzzy scalars (fuzzy points defined on the real- valued space R) to measure the distance between fuzzy points, which is consistent with the theory of fuzzy linear spaces in the sense of Xia and Guo (2003)[24] and hence more similar to the classical metric spaces.

The definitions in this chapter are different from the previous definition because fuzzy scalars are used instead of fuzzy numbers or real numbers to measure the distance between two fuzzy points. In this chapter fuzzy scalars are introduced in measuring the distances between fuzzy points.

Some basic concepts of fuzzy points and notations are presented below.

Fuzzy points are fuzzy sets x_λ of the following form [17].

$$X_a(y) = \begin{cases} \lambda , & y = x \\ 0 , & x \neq y \end{cases}$$

where $\lambda \in (0, 1)$.

In this chapter , fuzzy points are denoted by (x , λ) , and the set of all the fuzzy points defined on X is denoted by $\mathbf{P_F(x)}$ **,** where X is a nonempty set .

When $X = R$, fuzzy points are also called fuzzy scalars and the set of all the fuzzy scalars is denoted by $\mathbf{S_F(R)}$ **.** a fuzzy set A can be considerd as a set of fuzzy points belonging to it , i.e :

$$A = \{ (x , \lambda) : A(x) \geq \lambda \}$$

Or a set of fuzzy points on it ,

$$A = \{ (x , \lambda) : A(x) = \lambda \}$$

3.1 Fuzzy Metric

Definition 3.1.1

Let (x , λ) and (y , r) be two fuzzy scalars .

We define the following :

1) $(a , \lambda) + (b , r) = (a+b , \min \{ \lambda , r \})$

2) $(a , \lambda) \geq (b , r)$ means $a > b$ or $(a , \lambda) = (b , r)$

3) $(a, \lambda) > (b , r)$ means : $b \leq a$.

4) (a , λ) is said to be nonnegative if $a \geq 0$.

The set of all nonnegative fuzzy scalars is denoted by $\mathbf{S}^{+}_{F}(R)$.

Proposition 3.1.2

The orders defined in definition 3.1.1 (2) and definition 3.1.1 (3) are both partial orders .

proof:

(1) Transitive

from definition 3.1.1 (3)

$(a , \lambda) > (b , r) \Longrightarrow a \geq b$

also , $(b , r) > (c , \alpha) \Longrightarrow b \geq c$

so , $a \geq c$ since we deal with numbers $\longrightarrow (a , \lambda) > (c , \alpha)$.

(2) R**eflexive**

want to prove that : $(a , \lambda) \leq (a , \lambda)$

Now $(a , \lambda) = (a , \lambda)$, and from definition 3.1.1 (2) that is mean :

$(a , \lambda) \leq (a , \lambda)$.

(3) Antisymmetric

We want to prove : if $(a , \lambda) \leq (b , r)$ and $(b , r) \leq (a , \lambda)$ then :

$(a , \lambda) = (b , r)$.

Since $(a , \lambda) \leq (b , r)$ then $a < b$ or $(a , \lambda) = (b , r)$. *

Also $(b , r) \leq (a , \lambda)$ then $b < a$ or $(b , r) = (a , \lambda)$. **

From (*) and (**) $(a , \lambda) = (b , r)$.

Definition 3.1.3

Suppose X is a nonempty set define :

$$\mathbf{D : P_F(x) \times P_F(x) \longrightarrow S^{+}_{F}(R)}$$

satisfies :

1) non negative : $\mathbf{D}((x,\lambda) , (y,r)) = 0$ iff $x=y$ and $\lambda = r = 1$

2) symmetric : $\mathbf{D}((x,\lambda) , (y,r)) = \mathbf{D}((y,r),(x,\lambda))$.

3) Triangle inequality :

D((x , ª) , (z , p)) < **D**((x , ª) , (y , r)) + **D**((y , r) , (z , p))

For any (X , λ) , (y , r) , (z , p) ∈ $P_F(X)$

D is called a fuzzy metric defined on P_F (x) and **D**((x, ª) , (y , r)) is called a fuzzy distance between the two fuzzy points .

and **(P_F(x) , D)** is said to be a fuzzy metric space .

Now we look at few examples :

Example 3.1.4

Suppose (X ,d) is an ordinary metric space . the distance of any two fuzzy points (x , λ) , (y , r) in $\mathbf{P}_F(X)$ is defined by :

$$\mathbf{D}((x,\lambda),(y,r)) = (\mathbf{d}(x,y) , \min \{\lambda,r\})$$

Where **d**(x,y) is the distance between x and y defined in (X , d) .

Then ($\mathbf{P_F}(x)$, $\mathbf{D}$) is a fuzzy metric space .

Proof :

We can prove that **D** satisfies the three conditions in definition 3.1.2

1) non negative : suppose (x,λ) and (y,r) are two fuzzy points in $\mathbf{P}_F(x)$.

Since **d** is a metric , then $\mathbf{d}(x,y) \geq 0$. it is clear that :

$\mathbf{D}((x,\lambda) , (y,r)) = 0$ iff $\mathbf{d}(x,y) = 0$ and $\min \{ \lambda , r \} = 1$ which is equivelant to ($x=y$ and $\lambda=r=1$) .

2) symmetric : for any $\{ (x,\lambda) , (y,r) \} \subset \mathbf{P_F(x)}$ One have:

$$\mathbf{D}((x,\lambda) , (y,r)) = (\mathbf{d}(x,y) , \min \{ \lambda , r \})$$

$$= (\mathbf{d}(y,x) , \min \{ r , \lambda \})$$

$$= \mathbf{D}((y,r) , (x,\lambda)) .$$

3) triangle inequality: for any $\{ (x,\lambda) , (y,r) , (z , s) \} \subset \mathbf{P_F(x)}$ **,** we have:

$$\mathbf{D}((x,\lambda) ,(z,s)) = (\mathbf{d}(x,z) , \min \{ \lambda , s\})$$

$$< (\mathbf{d}(x,y) + \mathbf{d}(y,z) , \min \{ \lambda , s, r \})$$

$$= (\mathbf{d}(x,y) , \min \{\lambda,r\})+(\mathbf{d}(y,z) , \min\{r,s\}) .$$

$$= \mathbf{D} ((x,\lambda) , (y,r)) + \mathbf{D} ((y,r), (z,s)) .$$

Example 3.1.5

We denote R^2 the usual 2- dimensional Euclidean space . the distance between arbitrary two fuzzy points p = (x_1 , y_1) and Q = (x_2 , y_2) denoted by :

D (P_λ , Q_r) is defined by : (**d**(P,Q) , min { λ , r }) where

d (P ,Q) = $\sqrt{(x_2 - x_1)2 + (y_2 - y_1)2}$ **.**

proof :

let P = (x_1 , y_1) , Q = (x_2 , y_2) , R = (x_3 , y_3) then :

We can prove that **D** satisfies the three conditions in definition 3.1.2

1) non negative : suppose (P,λ) and (Q ,r) are two fuzzy points in $\mathbf{P}_F(x)$.

Since **d** is a metric , then **d**(P,Q) ≥ 0 . it is clear that :

D((P,λ) , (Q,r)) = 0 iff **d** (P,Q) = 0 and min { λ , r } = 1 which is equivelant to (P=Q and λ=r=1) .

2) symmetric : for any {(P,λ) , (Q,r)} $\subset$ $\mathbf{P_F(x)}$ One have:

D((P,λ) , (Q,r)) = (**d**(P,Q) , min { λ , r })

= (**d**(Q,P) , min { r , λ })

= **D**((Q,r) , (P,λ)) .

3) triangle inequality: for any { (P,λ) , (Q,r) , (R , s) } $\subset$ $\mathbf{P_F(x)}$ **,** we have:

$\mathbf{D}((P,\lambda),(R,s)) = (\mathbf{d}(P,R), \min\{\lambda, s\})$

$< (\mathbf{d}(P,Q) + \mathbf{d}(Q,R), \min\{\lambda, s, r\})$

$= (\mathbf{d}(P,Q), \min\{\lambda,r\}) + (\mathbf{d}(Q,R), \min\{r,s\})$.

$= \mathbf{D}((P,\lambda),(Q,r)) + \mathbf{D}((Q,r),(R,s))$.

Definition 3.1.6

suppose X is a nonempty set and $\mathbf{D} : \mathbf{P_F(x)} \times \mathbf{P_F(x)} \longrightarrow \mathbf{S_F^+(R)}$ is a mapping ($\mathbf{p_F(x)}$, $\mathbf{D}$) is said to be a strong fuzzy metric space if it satisfies the following conditions :

(1) non negative : $\mathbf{D}((x,\lambda),(y,r)) = 0$ iff $x=y$ and $\lambda = r = 1$

(2) symmetric : $\mathbf{D}((x,\lambda),(y,r)) = \mathbf{D}((y,r),(x,\lambda))$.

(3$^\backslash$) $\mathbf{D}((x,\lambda),(z,p)) \leq \mathbf{D}((x,\lambda),(y,r)) + \mathbf{D}((y,r),(z,p))$

Remark 3.1.7

Every strong fuzzy metric space is afuzzy metric space .

Because :

(1) non negative : $\mathbf{D}((x,\lambda),(y,r)) = 0$ iff $x=y$ and $\lambda = r = 1$ and

(2) symmetric : $\mathbf{D}((x,\lambda),(y,r)) = \mathbf{D}((y,r),(x,\lambda))$, are the same .

And (3$^\backslash$) which say : (3$^\backslash$) $\mathbf{D}((x,\lambda),(z,p)) \leq \mathbf{D}((x,\lambda),(y,r)) + \mathbf{D}((y,r),(z,p))$ implies (3) which says :

$\mathbf{D}((x, {}^{a}), (z, p)) < \mathbf{D}((x, {}^{a}), (y, r)) + \mathbf{D}((y, r), (z, p))$

For any $(X, \lambda), (y, r), (z, p) \in P_F(X)$.

3.2 The completeness of fuzzy metric space

In this section , we mainly consider the convergence of a sequence of fuzzy points and the completeness of induced fuzzy metric spaces. Since fuzzy scalars are used to measure the distances between fuzzy points , the convergence of a sequence of fuzzy scalars is considered first.

Definition 3.2.1

Let $\{(a_n, \lambda_n)\}$ be a sequence of fuzzy scalars . it is said to be convergent to a fuzzy scalar (a, λ) , $\lambda \neq 0$.

i.e $\lim\limits_{n\to\infty}(a_n, \lambda_n) = (a, \lambda)$ if $\lim\limits_{n\to\infty} a_n = a$. also $\{\lambda_i : \lambda_i \leq \lambda , i \in N\}$ is a finite set and there exists a subsequence of $\{\lambda_i\}$ denoted by $\{\lambda_t\}$, such that $\lim\limits_{n\to\infty} \lambda_n = \lambda$.

Example 3.2.2

Consider the sequence of fuzzy scalars $\{(a,0.3), (a, 0.51), (a,0.4), (a, 0.501), (a, 0.2), (a, 0.5001), (a,0.50001), \ldots\}$. this sequence is convergent since :

we can find a finite set say $\{\lambda_j\} = \{.3, .4, .2\}$.

also , we can find a subsequence of $\{\lambda_j\}$ say:

$\{ \lambda_n \} = \{ .51 , .501 , .5001, ..50001 , \ldots. \}$

such that :

$$\lim_{n\to\infty} \{ .51 , .501 , .5001, ..50001 , \ldots. \} = 0.5$$

Therefore the sequence $\{ (a_n , \lambda_n)\} \longrightarrow (a , \lambda) = (a , 0.5)$

Definition 3.2.3

Suppose ($\mathbf{P_F(X)}$, $\mathbf{D}$) is the induced fuzzy metric space of (x , d) and

$\{ (x_n , \lambda_n)\}$ is a sequance of fuzzy points in ($\mathbf{P_F(X)}$, $\mathbf{D}$) .

$\{ (x_n , \lambda_n)\}$ is said to be convergent to a fuzzy point (x , λ) ,if :

$$\lim_{n\to\infty} D((x_n , a_n) , (x , a)) = 0_a$$

And for any r ϵ (0 , 1] such that :

$$\lim_{n\to\infty} D ((x_n , a_n) , (x , r)) = 0_r$$

One has $\lambda \geq r$. (x , λ) is called the limit of the sequence , denoted by :

$$\lim_{n\to\infty} (x_n , \lambda_n) = (x , \lambda) .$$

Proposition 3.2.4

Suppose $\{(x_n , \lambda_n)\}$ is a sequence of fuzzy points in ($\mathbf{P_F(X)}$,$\mathbf{D}$) and (x,λ) ϵ ($\mathbf{P_F(X)}$, $\mathbf{D}$) , $\lambda \neq 0$. we have that :

$$\lim_{n\to\infty} (x_n , a_n) = (x , a) \text{ if and only if } \lim_{n\to\infty} x_n = x ,$$

$\{ a_j : a_j < a , j \in N \}$ is a finite set and there exists a

subsequence of $\{ a_j \}$ denoted by : $\{ a_n \}$, such that : $\lim\limits_{n \to \infty} a_n = a$

Definition 3.2.5

A sequence $\{ x_n \}$ of a points in a metric space (X , d) is called a Cauchy sequence if for each £ > 0 , there exists a positive integer **N** such that $\mathbf{d}(x_n , x_m) <$ £ whenever $n,m \geq N$.

Remark 3.2.6

If the sequence $\{ x_n \}$ in a metric space (X , d) is a Cauchy , then we shall write : $\lim\limits_{n \to \infty} \mathbf{d}(x_n , x_m) = 0$

Definition 3.2.7

A metric space (X ,d) is complete if every Cauchy sequence in (X , d) converges .

Definition 3.2.8

A sequence of fuzzy points $(x_n , a_n) \in (\mathbf{P_F(X)} , \mathbf{D})$ is said to be a cauchy sequence if there exists some $\lambda \in (0 , 1]$ such that :

$$\lim_{n \to \infty} \mathbf{D} ((x_{m+n} , \lambda_{m+n}) , (x_n , \lambda_n)) = 0_\lambda , \quad \forall \ m \in \mathbf{N}$$

Remark 3.2.9

Every Cauchy sequence of fuzzy points defined above has a unique fuzzy point as its limit .

Definition 3.2.10

An induced fuzzy metric space is said to be complete if any Cauchy sequence in it has a limit in the space .

3.3 Fuzzy topology spaces induced by fuzzy metric spaces .

In this section we provide a suitable method to construct a fuzzy topology of any ordinary metric space. To do this , the definition of fuzzy closed sets with respect to induced fuzzy metric space is given first . suppose (X , d) is an ordinary metric space . since a fuzzy set **A** in X can be represent as a set of fuzzy points belonging to it , **A** can be regarded as a subset of $\mathbf{P_F(X)}$ called a fuzzy set in the induced fuzzy metric space ($\mathbf{P_F(X)}$, **D**) in the following definition .

Definition 3.3.1

A fuzzy set **A** in ($\mathbf{P_F(X)}$, **D**) is said to be closed if the limit of any Cauchy sequence in **A** belongs to it . a fuzzy set **A** in ($\mathbf{P_F(X)}$, **D**) is said to be open if $\mathbf{A^C}$ is a fuzzy closed set , where : $\mathbf{A^C(X) = 1- A(X)}$, for any x ϵ X .

Remarke 3.3.2

We will show that every induced fuzzy metric space can induced a fuzzy topology .

Lemma 3.3.3[17]

Any subsequence of a Cauchy sequence of fuzzy points is also a Cauchy sequence and has the same limit as the original one .

Theorem 3.3.4

Suppose (**P_F(X)** , **D**) is the induced fuzzy metric space of a metric space (x , d) . Then (X , T_F) is a fuzzy topology space in the sense of p_u (1980)[17] , called the fuzzy topology space induced by:

(**P_F(X)** , **D**) , where T_F Is defined by :

T_F = { A $\subset$ **P_F(X)** : A is a fuzzy closed set in (**P_F(X)** , **D**) . }

Proof :

T_F satisfies the three conditions in the definition of fuzzy topology as follows :

1) it is clear that X and Φ are fuzzy closed sets .

2) For any {A, B} $\subset$ T_F , we prove in the following that A $\cup$ B ϵ T_F . for any Cauchy sequence of fuzzy points { (y_n , $¥_m$) } included in A $\cup$ B , A or B , say A , must contain a subsequence { (y_m , $¥_m$) } of

{ (y_n , $¥_n$) } . (lemma 3.3.1)

{ (y_m , $¥_m$) } is also a Cauchy sequence and hence has a limit . since A is a closed fuzzy set , the limit of { (y_m , $¥_m$) } which is also the limit of { (y_n , $¥_n$) } is included in A . in consequence , the limit of

$\{ (y_n , ¥_n) \}$ is included in $A \cup B$, which implies that $A \cup B \in \tau_F$.

3) for any $\{ A_i \} \subset \tau_F$, where I is an arbitrary index set , it only need to be proved that $\cap A_i \in \tau_F$ .for any Cauchy sequence in $\cap A_i$, denoted by $\{ (x_n, a_n)\}$

We have that $\{ (x_n , a_n) \} \subset A_i$ for any $i \in I$. since every A_i is a closed fuzzy set , the limit of $\{(x_n , a_n)\}$ is in A_i for any $i \in I$. It follows that $\cap A_i$ is a closed fuzzy set in the sense of definition 3.3.1. Therefore $\cap A_i \in \tau_F$.

Chapter Four

Some results of metric spaces and Fuzzy metric spaces

Chapter Four

Some results of metric spaces and Fuzzy metric spaces

Introduction

In this chapter , we will deal with the concept of fuzzy metric spaces as well as some properties and applications in fixed point theory under fuzzy setting .

The problem of constructing an interesting theory of fuzzy metric spaces has been suggested by several authers . some of them modified the classical definition of metric spaces . others put new independent definitions but it turns out that they are really an extension of the classical definition . so it would be wise to present the concepts of classical metric spaces in the first section of this chapter (section 4.1) followed by the extention of the definition and properties to fuzzy metric spaces .

4.1 Metric spaces

we defined a metric space on a set X to be the orderd pair (X , d) where X is a set , and d is a non-negative real valued function on the set X x X that is symmetric , satisfies the triangle inequality

$d^{-1}(\{0\}) = \{ (x ,x): x \in X \}$. if this last condition is replaced by

$d^{-1}(\{0\}) \supseteq \{ (x , x) : x \in X \}$ then the pair (X , d) is called p psedo metric space .

we will present now some examples :

Example 4.1.1

Let $X = R^{m,}$ define a function d : $R^m \times R^m \to R$

By $d(P, Q) = \max_{1 \le i \le m} |x_i - y_i|$; where $P = (x_1, x_2, \ldots, x_m)$

$Q = (y_1, y_2, \ldots, y_m)$

The function d is usually called the max metric on R^m and denoted by d_∞ and(X, d_∞) is a metric space .

Example 4.1.2

let $X = R$. define the function $d : X \times X \to R$ by $d(x,y) = (x - y)^2$, then d is not a metric and (X , d) is not a metric space.

Definition 4.1.3

let (X , d) be a metric space and r be a positive real number. the **open ball** in (X , d) of radius r centered at $a \in X$ is defined by:

$B(a, r) = \{ y \in X : d(a, y) < r \}$. and a is called the center of the open ball .

The open ball B(a,r) is also called a neighberhood of the point $a \in X$.

It is clear that for any two open balls of the same center , one of them Should be Contains the other .

Definition 4.1.4

let (X , d) be a metric space , $\{ x_n \}$ be a sequence in X .

The sequence $\{ x_n \}$ converges to a point a in X if for each $\varepsilon > 0$ there is a positive integer N such that $d (x_n , a) < \varepsilon$, whenever $n \geq N$.

and we write $x_n \longrightarrow a$

observe that $x_n \longrightarrow a$ is equivallent to :

$\lim\limits_{n\to\infty} d (x_n , a) = 0$.

Definition 4.1.5

Let (X , d) be a metric space and **A** be a subset of X .

A point x ϵ X is called (**a limit point**) of **A** if each open ball with the center x contains at least one point of **A** different from x , that is

$\{ B (x , r) - \{x\} \} \cap A \} \neq \Phi$; for every r >0 we say x is a cluster point of A if each open ball with center x as a nonempty intersection with A .

Remark 4.1.6

The set of all limit points of **A** is called the **derived set** of **A** and denoted by $A^{\backslash}$ and the set of all cluster point of A is called the closure of A and denoted by A .

Consider now , the following example :

Example 4.1.7

Let X = R and define a metric on X by $d(x,y) = |x - y|$, which is called the usual metric on R . the sequence $\{\frac{1}{2n} : n \in N\}$ is a sequence in R and converges to 0 in R .

Remark 4.1.8

In a metric space (X , d) , if a sequence $\{x_n\}$ converges to a . then a is a cluster point , but the converse is not true . the following example explains:

Example 4 .1.9

Consider the sequence $\{ (-1)^n : n \in N \}$ in (R , usual metric) .

Then -1 and 1 are the cluster points of $\{ X_n \}$.

But the sequence $\{ (-1)^n : n \in N \}$ does not converge .

Definition 4.1.10

A sequence $\{ x_n \}$ of a points in a metric space (X , d) is called a **Cauchy sequence** if for each $\varepsilon > 0$, there exists a positive integer k such that $d (x_n , x_m) < \varepsilon$ whenever $n,m \geq k$.

Remark 4.1.11

It is clear that , in any metric space (X ,d) , if a sequence $\{x_n\}$ is a Cauchy sequence then $\lim_{n,m\to\infty} d(x_n , x_m) = 0$.

It is obvious that every convergent sequence is a cauchy sequence .

Definition 4.1.12

A metric space (X , d) is **complete** if every Cauchy sequence in (X , d) converges .

Example 4.1.13

let X =R be equipped with the usual metric d and consider the sequence { x_n} in X by :

$\{ 1 - \frac{1}{3^n} : n \in N \}$

Then $\lim\limits_{n,m\to\infty} d(x_n , x_m) = \lim\limits_{n,m\to\infty} (\frac{1}{3^n} - \frac{1}{3^m}) = 0$.

hence the sequence $\{ 1 - \frac{1}{3^n} : n \in N \}$ is a Cauchy sequence .

we provide an example of a sequence in a metric space (X , d) which is cauchy sequence .

Example 4.1.14

In (R , usual metric) ;

Consider the sequence { x_n } where :

$X_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{(-1)^{n+1}}{n}$. for n ϵ N

This sequence is a cauchy sequence because if n >m ≥ k ;

$d (x_n , x_m) = \left| \frac{1}{m+1} - \frac{1}{m+2} + \ldots \pm \frac{1}{n} \right| < \frac{1}{m} \leq \frac{1}{k}$

And rerefore , d (x_n , x_m) <£ whenever $k > \frac{1}{£}$

Observe that not every Cauchy sequence in a metric space converges. this is illustrated by an example below .

Example 4.1.15

Let X = (0 , 1) be equipped with the usual metric d . then the sequence

$\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \}$ in X is Cauchy but does not converge to a point in X .

This shows that a metric space (X , d) is not complete .

Definition 4.1.16

Let (X_1 , d_1) and (X_2 , d_2) be metric spaces and let P = (x_1 , x_2)

Q = (y_1 , y_2) be arbitrary points in the product X = X_1 x X_2 .

define :

$d (P , Q) = \max \{ d_1(x_1 , y_1) , d_2 (x_2 , y_2) \}$.

then d is a metric on X and (X , d) is called :

the **product of the metric spaces** (x_1 , d_1) and (x_2 , d_2) .

Definition 4.1.17

Let (x_n , d_n) , n = 1,2 ,…. be metric spaces , $X = \prod_{n=1}^{\infty} X_n$. then define the metric d on X ; $d (P , Q) = \sum_{n=1}^{\infty} 2^{-n} d_n(x_n, y_n)$,

where P = { x_1,x_2,x_3,…}

And Q = { y_1,y_2,y_3,…} are in X . (X , d) is a complete metric space if and only if each (X_n , d_n) , n = 1 , 2 , … is complete .

Definition 4.1.18

A subset A of a metric space (X , d) is said to be **open** if given any point $x \in A$, there exists $r > 0$ such that $B (x , r) \subseteq A$. and we say a set B is closed if B^C is open .

There are several ways to characterize closed sets in metric spaces the following are among them :

*a subset A in a metric space is closed if it contains all its limit points .

* a subset A in a metric space is closed if A equals the set of cluster points of A .

The following are always true : in any metric space (X,d) each open ball is an open set .

Definition 4.1.19

The sets of closure set and derived set of A are combined by the relation : $\overline{A} = A \cup A^{\backslash}$.

Definition 4.1.20

Let (X , d) be a metric space and let A be a nonempty subset of X . we say that A is **bounded** if there exist a positive real number M such that :

$d (x , y) \leq M$, for all $x , y \in A$.

if A is bounded , we define the **diameter** of A as :

dia (A) = sup { d (x , y) : x , y ϵ A } . if A is unbounded , we write dia (A) = ∞ .

if A = Φ we write dia (A) = 0 .

Definition 4.1.21

A subset A of a metric space (X , d) is said to be :

* rare (or no where dense) in X if it is closure Ã has no interior points .

* meager (or of first category) in X , if A is the union of the countable many sets each of which is rare in X .

* non – meager (or of second category) in X if A is not meager in X .

Definition 4.1.22

Let (X , d) and (Y , p) be two metric spaces a mapping :

f : (X , d) $\longrightarrow$ (Y , p)is an **isometry** if : p (f(x) , f(y)) = d (x , y)

For all x , y ϵ X . the metric space (x , d) is said to be isometric to the metric space (y , p) when there exists some isometry from (X , d) into (Y , p) .

Definition 4.1.23

Let (X , d) be a metric space . a metric space ($X^{\backslash}$, $d^{\backslash}$) is said to be a **completion** of the metric space (X , d) if ($X^{\backslash}$, $d^{\backslash}$) is complete and (X , d) is isometric to a dense subset of ($X^{\backslash}$, $d^{\backslash}$) .

Example 4.1.24

(R , usual metric) is a completion of (Q , usual metric) .

Theorem 4.1.25

Every metric space (X , d) has a completion and any two completions of (X , d) are isometric to each other .

Remark 4.1.26

In other words , up to isometry , there exists a unique completion of any metric space .

Definition 4.1.27

Let (x , d) be a metric space and A_1 , A_2 , ….. be a sequence of sets.

Then A_1 , A_2 , ……. Is said to be **nested** if $A_1 \supset A_2 \supset A_3 \supset$ …..

Theorem 4.1.28

Every nested sequence of nonempty closed sets with metric diameter zero has nonempty intersection .

4.2 Continuity and Uniform Continuity in Metric Spaces

Definition 4.2.1

Let (X , d) be a metric space , and let $T \subseteq X$. we say T is bounded if there exists $r > 0$ such that $T \subseteq B(a,r)$ for some a in X .

Bolzano – weierstrass theorem :

If T is a bounded subset of R^n with infinitely many points then T has at least one limit point a , a $\in R^n$.

Definition 4.2.2

Let $\hat{G}$ be a collection of open sets in the metric space (X , d) . we say $\hat{G}$ is an open cover for A $\subseteq$ X if $\bigcup_{G \in \hat{G}} G \supseteq A$.

Definition 4.2.3

We say the metric space (X , d) is compact if every open cover of X has a finite subcover .

Remark 4.2.4

An equivalent definition for a metric space to be compact is the following :

For any sequence { F_i } of closed subsets in (X , d) , if the intersection of any finite subfamily is not empty then the intersection of the elements of the sequence is not empty .

Proposition 4.2.5

Every compact subset A of a metric space(X , d) is bounded .

One of the most important properties of a closed and bounded interval in R when equipped with the usual metric is given in the next theorem .

Theorem 4.2.6

Heine Borel theorem

Let T be a closed and bounded subset of (R^n, usual metric) then T is compact .

We provide examples on (R , usual metric) where theorem is not applied .

Example 4.2.7

In (R , usual metric) , T_1 = [0 , ∞) is closed but not bounded . it is not compact ; because the open cover { (-1 , n) : n $\in$ N } has no finite subcover , and T_2 = (0 , 1) is bounded but not closed . it is not compact by taking the open cover { ($\frac{1}{n+1}$, 1) : n $\in$ N } has no finite subcover .

Let X be any infinite set and d be the discrete metric then (X , d) is not compact . because take the open cover { B(x , $\frac{1}{2}$) : x $\in$ X } this open cover has no finite subcover . observe here that (X , d) is bounded .

Definition 4.2.8

Let (X , d) be a metric space we say that D $\subseteq$ X is a dense if $\overline{D}$ = X

Definition 4.2.9

Let (X , d) be a metric space . if there is a countable dense subset in (X , d) then (X , d) is said to be **separable** .

We now provide an example of a metric space which is not separable.

Example 4.2.10

Let X denote the infinite set and d be discrete metric . then the metric space (X , d) is not separable .

Definition 4.2.11

A space X is a T_2 – space (Haus dorff space) iff whenever x and y are distinct points of X , there are disjoint open sets u and v in X with $x \in u$ and $y \in v$.

Proposition 4.2.12

Every metric space is a Housdorff space .

Proof:

Let (X , d) be a metric space , let $x , y \in X$ with $x \neq y$, since $x \neq y$

$d (x , y) = r > 0$, take $u = B (x , \frac{r}{3})$, $v = B (y , \frac{r}{3})$ then $u \cap v = \emptyset$ and (X , d) is Hausdorff .

Definition 4.2.13

Let (X , d) and (Y , p) be metric spaces .

The function $f : (X , d) \longrightarrow (Y , p)$ is said to be **continuous** at the point $x_0 \in X$ if for each $\varepsilon > 0$ there exists a ¢ > 0 such that : $p (f(x) ,f(x_0)) < \varepsilon$

Whenever $d(x, x_0) < ¢$.

We shall say that $f : (X, d) \longrightarrow (Y, p)$ is continuous if it is continuous at every $x \in X$.

Now, we present an equivalent definition of continuity using sequences.

Theorem 4.2.14

Let (X, d) and (Y, p) be metric spaces . then the following statements are equivalent :

* $f : (X, d) \longrightarrow (Y, p)$ is continuous .

**For a sequence $\{x_n\}$ and a point x in (X, d)

$\lim_n p(f(x_n), f(x)) = 0$, whenever $\lim_n d(x_n, x) = 0$.

An example of an identity function that is not continuous .

Example 4.2.15

Let $f : (R, \text{ usual metric }) \longrightarrow (R, \text{ discrete metric })$

Where $f(x) = x$ for each $x \in R$. then f is not a continuous function . to see this , take a sequence $\{\frac{1}{n} : n \in N\}$, then :

$\lim_n d(x_n, 0) = 0$, But $\lim_n p(f(x_n), f(0)) = 1$.

Definition 4.2.16

Let (X , d) and (Y , p) be two metric spaces .

The function f : (X , d) $\longrightarrow$ (Y , p) is **uniformly continuous** on X if and only if for every $\epsilon > 0$ there exists a ¢ > 0 such that if $x_1 \in X$, $x_2 \in X$ and $d (x_1 , x_2) <$ ¢ , then $p (f(x_1) , f(x_2)) < \varepsilon$.

We provide an example of a uniformly continuous function .

Example 4.2.17

Let X = [0 , 1] be equipped with the usual metric d and Y = R be equipped with the usual metric p . consider the function :

f : [0 , 1] $\longrightarrow$ R given by :

f (x) = 2x +1 for x $\in$ [0 , 1] . clearly f is uniformly continuous on [0 , 1] .

Next we provide an example of a function which is not uniformly continuous .

Example 4.2.18

Let X = R , Y = R be equipped with the usual metrics and

f : R $\longrightarrow$ R Given by $f(x) = x^2$. then f is not uniformly continuous .

4.3 Fuzzy Metric Space

In this section the concept of fuzzy metric space we deal with is due to Geoge and veeramani [5] and the axiomatic of this theory is explained as follows .

Definition 4.3.1 [5]

Let $\sigma : [0,1] \times [0,1] \longrightarrow [0,1]$ be a continuous function satisfies the following conditions :

1. $\sigma(r,t) = \sigma(t,r)$.
2. $(\sigma(r, \sigma(s,t)) = \sigma(\sigma(r,s),t)$
3. $\sigma(t,1) = t$
4. $t \le r \;, s \le m \longrightarrow \sigma(t,s) \le \sigma(r,m)$

A fuzzy metric space is an ordered triple (X , D , σ) . such that X is a (nonempty) set , σ is a continuous s – norm and D is a fuzzy set in $X \times X \times (0, \infty)$ satisfying the following conditions :

1. $D(x,y,t) > 0$
2. $D(x,y,t) = 1$ if and only if $x = y$
3. $D(x,y,t) = D(y,x,t)$
4. $\pm (D(x,y,t), D(y,z,s)) \le D(x,z,t+s)$
5. $D(x,y,\_) : (0,\infty) \longrightarrow [0,1]$ is continuous .

Whenever $x , y , z \in X$. $s , t \in (0,\infty)$.

Remarke 4.3.2

The axiom 2 is equivalent to the following :

D (x , x , t) = 1 for all x ϵ X and t > 0 , and D (x , y , t) < 1

for all x $\neq$ y , and t > 0 .

Now we illustrate the previous definition by the following examples:

Example 4.3.3 [5]

Let (X , d) be a metric space . denoted by (a . b) the usual product for all a , b ϵ [0, 1] , and let D_d be the fuzzy set defined on

X x X x (0 , ∞) by :

$D_d = \frac{t}{t + |y - x|}$ is called standard fuzzy metric space .

Example 4.3.4 [18]

Let (X , d) be a metric space , and B(x , r) the open ball centered in x ϵ X With radius r > 0 , then for each n ϵ N , (X , D , $\wedge$) is a fuzzy metric space where D is given by :

$D(x , y , t) = \frac{1}{e^{\frac{|y-x|}{t}}}$,for all x,y ϵ X , t > 0. D is a fuzzy metric space.

Example 4.3.5 [21]

Let X= R^+ . define for x , y ϵ X , t > 0

$D(x , y , t) = \frac{\min\{ x ,y \}+ t}{\max\{ x ,y \}+ t}$

D (x , y , t) is a fuzzy metric space .

Definition 4.3.6

A fuzzy metric D on X is said to be stationary [8] , if D does not depend on t , i.e if for each x , y ϵ X , the function $D_{x,y}(t) = D(x, y, t)$ is constant . in this case we write

$D(x, y)$ instead of $D(x, y, t)$ and $B_M(X, \varepsilon)$ instead of $B_M(X, \varepsilon, t)$

Where $B_M(X, \varepsilon, t) = \{ y \in X : D(x, y, t) > 1 - \varepsilon \}$ for all $x \in X$, $\varepsilon \in (0, 1)$ and $t > 0$.

Definition 4.3.7

A subset A of X is said to be F – bounded if there exist $t > 0$ and

$r \in (0, 1)$ such that $D(x, y, t) > 1 - r$, for all $x , y \in A$.

Proposition 4.3.8

If (X , d) is a metric space , then : $A \subset X$ is bounded in (X , d) if and only if it is F – bounded in $(X, D_d, \pm)$.

Definition 4.3.9

A topological space (X, τ) is said to be fuzzy metrizable if there exists a fuzzy metric D on X compatible with τ , i.e $\tau_D = \tau$.

Definition 4.3.10

A sequence $\{ x_n \}$ in a fuzzy metric space $(X, D, \pm)$ is called a Cauchy sequence .

(or D – cauchy) , if for each $\varepsilon \in (0, 1)$, $t > 0$ there exists $n_0 \in N$ such that:

$D(x_n, x_m, t) > 1 - \varepsilon$, for all m , $n \geq n_0$.

Proposition 4.3.11

$\{x_n\}$ is a d – Cauchy sequence , i.e a Cauchy sequence in (X , d) if and only if it is a Cauchy sequence in (X , D_d , ±)

Definition 4.3.12

Let (X , D , ±) be a fuzzy metric space , a sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} D(x_n, x, t) = 1$, for all $t > 0$.

Remark 4.3.13

A metrizable topological space (X , τ) is said to be completely metrizable if it admits a complete metric . On the other hand , a fuzzy metric space (X , D , ±) is called complete if every Cauchy sequence is convergent . if (X , D , ±) is a complete fuzzy metric space , we say that D is a complete fuzzy metric on X .

Theorem 4.3.14

Let (X , D , ±) be a complete fuzzy metric space . then (X , τ_D) is completely metrizable .

4.4 Continuity and uniform continuity

Definition 4.4.1

A mapping from X to Y is said to be uniformly continuous if for each $\varepsilon \in (0, 1)$, and each $t > 0$, there exist $r \in (0, 1)$ and $s > 0$, such that:

$L(f(x), f(y), t) > 1 - \varepsilon$. Whenever $D(x, y, s) > 1 - r$.

Definition 4.4.2

We say that a real valued function f on the fuzzy metric space $(X, D, \pm)$ is R – uniformly continuous provided that for each $\varepsilon > 0$, there exist $r \in (0, 1)$ and $s > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $D(x, y, s) > 1 - r$.

Definition 4.4.3

A fuzzy metric $(D, \pm)$ on a set X is called equinormal if for each pair of disjoint nonempty closed subsets A and B of (x, τ_D) there is $s > 0$ such that :

$\operatorname{Sup}\{D(a, b, s) : a \in A, b \in B\} < 1$.

Theorem 4.4.4 [18]

For a fuzzy metric space $(X, D, \pm)$ the following are equivalent:

1. for each fuzzy metric space $(Y, L, \underline{\pm}^{\backslash})$ any continuous mapping from (x, τ_D) to (Y, τ_L) is uniformly continuous as a mapping

From (X , D , $\underline{+}^{\backslash}$) to (Y , L , $\underline{+}^{\backslash}$) .

2. every real valued continuous function on (x , τ_D) is R – uniformly continuous on (X , D , ±) .

3. (D , ±) is an equinomal fuzzy metric on X .

Definition 4.4.5

A mapping **f** from a fuzzy metric space (X , D) to a fuzzy metric space (Y , L) is called t – uniformly continuous if for each $\varepsilon \in (0 , 1)$ and each $t > 0$, there exists $r \in (0 , 1)$ such that

$L (f(x) , f(y) , t) > 1 - \varepsilon$, whenever $D (x , y ,t) > 1 - r$.

Remark 4.4.6 [7]

Every continuous mapping from a compact fuzzy metric space to a fuzzy metric space is uniformly continuous .

Proposition 4.4.7

Every continuous mapping from a compact fuzzy metric space

(X , D , ±) to a fuzzy metric space (Y , L , ±) is t – uniformly continuous .

Definition 4.4.8

A fuzzy metric (D , ±) on a set X is called t – equinormal if for each pair of disjoint nonempty closed subsets A and B of (x , τ_D) and each

$t > 0$, sup { D (a , b , t) : $a \in A$, $b \in B$ } < 1 .

Theorem 4.4.9

For a fuzzy metric space (X , D , ±) the following are equivalent :

1. for each fuzzy metric space (Y , L , •) any continuous mapping from (x, τ_D) to (y , τ_L) is t – uniformly continuous as a mapping from

(X , D , ±) to(Y , L , •) .

2. the fuzzy metric (D , ±) is t – equinormal .

4.5 On completion of fuzzy metric spaces

Given a metric space (X , d) we shall denote by (X' , d') the

(metric completion) of (X , d) .

In a first attempt to obtain a satisfactory idea of fuzzy metric completion we start by analyzing the relationship between the standard fuzzy metrics of d and d' , respectively .

Example 4.5.1

Let (X , d) be a metric space and let f be an isometry from (X , d) onto a dense subspace of (X' , d') .the standard fuzzy metric

($D_{d'}$, .) of d is given by :

$D_{d'}(x' , y' , t) = \frac{t}{t+d'(x',y')}$.

Also we have $f : x \rightarrow D$ where $\bar{D}$ = X' , for all x' , y' ϵ X' and t >0 , hence , we have :

D_d (x , y , t) = $D_{d'}$ = (f(x) , f(y) , t) for all x , y ϵ X and t > 0 .

Definition 4.5.2

Let (X , D , ±) and (Y , L , •) be two fuzzy metric spaces .

A mapping f from X to Y is calld an isometry if for each x , y ϵ X and each t > 0 , D (x , y , t) = L (f(x) , f(y) , t) .

Definition 4.5.3

Two fuzzy metric spaces (X , D , ±) and (Y , L , •) are called isometric if there is an isometry from X onto Y .

Definition 4.5.4

Let (X , D , ±) be a fuzzy metric space . A fuzzy metric completion of (X , D,±) is a complete fuzzy metric space (Y , L , •) such that

(X , D , ±) is isometric to a dense Subspace of y .

Next , we show that there exists a fuzzy metric space that does not admit any fuzzy metric completion .

Example 4.5.5 [10]

Let $(x_n)_{n=3}^{\infty}$ and $(y_n)_{n=3}^{\infty}$ be two sequences of distinct points such that A ∩ B = Φ , where A = { X_n : n ≥ 3 } and B = { Y_n : n ≥ 3 } .

Put $X = A \cup B$. Define a real valued function « on X x X x (0 , ∞) as follows :

$D(x_n, x_m) = D(y_n, y_m) = 1 - [\frac{1}{n \wedge m} - \frac{1}{n \vee m}]$,

$D(x_n, y_m) = D(y_m, x_n) = \frac{1}{n} + \frac{1}{m}$, for all n , m $\geq$ 3 . then

(X , D , ·) a stationary fuzzy metric space .

Now , we want to talk about a characterizing completable fuzzy metric spaces .

Definition 4.5.6

Let (X , D , ±) be a fuzzy metric space . then a pair a_n , b_n of cauchy sequences in X , is called :

a. point – equivalent if there is s > 0 such that $\lim_{n\to\infty} D(a_n, b_n, s) = 1$

b.equivalent , denoted by $a_n \sim b_n$, if $\lim_{n\to\infty} D(a_n, b_n, t) = 1$ for all t > 0 .

Theorem 4.5.7

A fuzzy metric space (X , D , ±) is completable if and only if it satisfies the two following conditions :

1. given two Cauchy sequences $(a_n)_n$, $(b_n)_n$ in X , then $\lim_{t\to n} D(a_n, b_n, t)$ is a continuous function on (0 , ∞) with values in (0 , 1] .

2. each pair of point – equivalent Cauchy sequences is equivalent .

Definition 4.5.8 [15]

Let (X , D) be a fuzzy metric space . a sequence { X_n } in X is said to be point Convergent to $x_0 \in X$ if $\lim_{n\to\infty} D(x_n , x_0 , t_0) = 1$, for some $t_0 > 0$.

Remark 4.5.9 [15]

In such a case we say that { X_n } is p – convergent to x_0 for all $t > 0$.

Now , the following properties hold :

1. if $\lim_{n\to\infty} D(x_n , x , t_1) = 1$ and $\lim_{n\to\infty} D(x_n , y , t_2) = 1$ then $x = y$

2. if $\lim_{n\to\infty} D(x_n , x_0 , t_0) = 1$ then $\lim_{k\to\infty} D(x_{nk} , x_0 , t_0) =1$ for each subseuence (X_{nk}) of { X_n } .

An example of a p – convergent sequence which is not convergent is given in the next example .

Example 4.5.10 [15]

Let $\{ x_n \} \subset (0 , 1)$ be a strictly increasing sequence convergent to 1 respect to the usual topology of R and $X = \{ X_n \} \cup \{ 1 \}$ define on

$X^2 \times R^+$ the function D given by :

$D(x , x , t) =1$ for each $x \in X$, $t > 0$, $D(x_n , x_m , t) = \min \{ x_n , x_m \}$,

for all $m , n \in N$, $t > 0$, and $D(x_n , 1 , t) = D(1 , x_n , t) =$

min { x_n , t } , for all n ϵ N , t > 0 . then (D , ±) is a fuzzy metric on X , where a* b = min { a , b } . the sequence { X_n } is not convergent since $\lim_{n\to\infty} D(X_n, 1, \frac{1}{2}) = \frac{1}{2}$. nevertheless it is p – convergent to 1 , since

$\lim_{n\to\infty} D(X_n, 1, 1) = 1$.

4.6 Principal fuzzy metrics

Definition 4.6.1

We say that the fuzzy metric space (X , D , ±)

Is principal if {B(X, r, t) : r ∈ [0 , 1] is a local base at x ∈ X , for each x ∈X and each t > 0 .

Theorem 4.6.2

The fuzzy metric space (X , D) is principal if and only if all p –convergent sequences are convergent .

Definition 4.6.3

Let (X , D) be a fuzzy metric space . A sequence { x_n } in X is said to be p – Cauchy if for each ε ϵ (0 , 1) there are n_0 ϵ N and $t_0 > 0$ such that: $D(x_n, x_m, t_0) > 1 - \varepsilon$ for all n, m $\geq n_0$, i.e $\lim_{m,n} D(x_n, x_m, t_0) = 1$ for some $t_0 > 0$.

Remark 4.6.4

{ x_n } is a Cauchy sequence if and only if { x_n } is p – Cauchy for all t > 0 and it is clear that , p – convergent sequences are p – Cauchy .

Definition 4.6.5

The fuzzy metric space (X , D) is called p – complete if every

p – Cauchy sequence in X is p – convergent to some point of X . in this case D is called p – complete .

Proposition 4.6.6

Let (x , D) be a principal fuzzy metric space . if X is p – complete then X is complete .

Definition 4.6.7

F is said to be continuous at $x_0 \in X$ if given $\varepsilon \in (0 , 1)$ and $t > 0$, there exist ¢ $\in (0 , 1)$ and $s > 0$, such that $D (x_0 , x , s) > 1 -$ ¢ implies

$L (f(x_0) , f(x) , t) > 1 – \varepsilon$.

Definition 4.6.8

We will say that a mapping f from the fuzzy metric space (x , D) to a fuzzy metric space (Y , L) is t – continuous at $x_0 \in X$ if given $\varepsilon \in (0 , 1)$ and $t > 0$ there exists ¢ $\in (0 , 1)$ such that $D (x_0 , x , t) > 1 -$ ¢ , implies $L (f(x_0) , f(x) , t) > 1 – \varepsilon$.

Remark 4.6.9 [11]

We say that f is t – continuous on X if it is t – continuous at each point of X.

If D is a stationary fuzzy metric then each continuous mapping is t – continuous and clearly if f is t – continuous at x_0 then f is continuous at x_0 . The converse is false . [11]

Remark 4.6.10

It is clear that each t – uniformly continuous mapping is t – continuous and the converse is not true .

Example 4.6.11

This is example of a t – continuous mapping which is not t – uniformly continuous .

Let X = { 1 , 2 , 3 , …. } .consider on X the fuzzy metric D for the usual product, given by :

$$D(m, n, t) = \begin{cases} \dfrac{\min\{m\ n\}}{\max\{m, n\}} \cdot t, & m \neq n, \ t < 1 \\[2ex] \dfrac{\min\{m, n\}}{\max\{m, n\}}, & \text{else where} \end{cases}$$

τ_D is the discrete topology on X .

Now , consider the mapping f : X $\rightarrow$R defined by :

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is odd} \\ 0 & \text{if } x \text{ is even} \end{cases}$$

Consider the fuzzy metric D on X and the standard fuzzy metric $D_{|.|}$ on R . we will see that f is not t – uniformly continuous for these fuzzy metrics .

Let t = 1 and ε = 0.5 . for every ¢ ϵ (0 , 1) there exist n ϵ X such that $\frac{n}{n+1} > 1$ - ¢ .

And so D (n , n+1 , t) = $\frac{n}{n+1} > 1 - \delta$, therefor

D|.| = (f(n) , f(n+1) , t) = $\frac{1}{1+1} = \frac{1}{2}$

And so f is not t – uniformly continuous .

Proposition 4.6.12

Let f be a mapping from the fuzzy metric space (X , D) to the fuzzy metric space (Y , L) , continuous at x_0 . if D is principal then **f** is t – continuous at x_0 .

4.7 Fixed point theorems in fuzzy metric spaces

Introduction

The concept of a fuzzy set was introduced by zadeh [26] in 1965 . this concept was used in topology and analysis by many authors . George and veeramani [6] modified the concept of fuzzy metric space introduced by kramosil and michalek [14]and defined the hausdorff topology of fuzzy metric spaces , which have important applications [20] in quantum particle physics .

The aim of this section is to obtain fixed point of mapping satisfying an implicit relation on fuzzy metric spaces .

First, we have defined in previous sections of this chapter the continuous t – norm which satisfies 4 conditions and we also defined a fuzzy metric space (x , D , ±) such that x is a non empty set ,± is S – norm and D is a fuzzy set on X x X x (0 , ∞) satisfying five conditions . and we have defined a compact fuzzy metric space and complete fuzzy metric space .

And we said that a sequence { X_n } is said to be converge to x in X , if and only if

$\lim_{n\to\infty} D(x_n , x , t) = 1$ for all $t > 0$.

Also a sequence { X_n } in X is an D – Cauchy sequence if and only if for each ε ϵ (0 , 1) , t > 0 , there exists n_0 ϵ N such that

$D (x_m , x_n , t) > 1 - \varepsilon$ for any m , n $\geq n_0$.

Definition 4.7.1

Let (x , D , ±) be a fuzzy metric space . for t > 0 , the open ball

B (x , r , t) with a center x ϵ X and a radius 0 < r < 1 is defined by:

$B (x , r , t) = \{ y \in X , D (x , y ,t) > 1 - r \}$.

Remark 4.7.2

A subset A ⊂ X is called open if for each x ϵ A , there exist t > 0 and 0 < r < 1 such that B(x , r , t) ⊂ A . Let τ denote the family of all open subsets of X . then , τ is called the topology on X induced by the fuzzy metric D . This topology is hausdorff . a subset A of X is said to be F – bounded if there exist t > 0 and 0 < r < 1 such that:

$D(x, y, t) > 1 - r$ for all $x, y \in A$.

Example 4.7.3

Let $X = R$. denote $a * b = a\,b$. for all $a, b \in [0, 1]$.

For each $t \in (0, \infty)$, define :

$D(x, y, t) = \dfrac{t}{t+|x-y|}$, for all $x, y \in X$. then , (x , D , ±) is a fuzzy metric space .

Definition 4.7.4

Let (x , D , ±) be a fuzzy metric space . D is said to be continuous on

$X \times X \times (0, \infty)$ if $\lim_{n\to\infty} D(x_n, y_n, t_n) = D(x, y, t)$

Whenever $\{(x_n, y_n, t_n)\}$ is a sequence in $X \times X \times (0, \infty)$, that is :

$\lim_{n\to\infty} D(x_n, x, t) = \lim_{n\to\infty} D(y_n, y, t) = 1$

$\lim_{n\to\infty} D(x, y, t_n) = D(x, y, t)$

Now , consider (x , D , ±) be a fuzzy metric space and $s \neq \Phi \subseteq X$.

Define ₵$_D(s, t) = \inf\{D(x, y, t) : x, y \in s\}$ for all $t > 0$. for an

$A_n = \{x_n, x_{n+1}, \ldots\}$ in fuzzy metric space (x , D , ±) ,

Let $r_n(t) =$ ₵$_D(A_n, t)$, then $r_n(t)$ is finite for all $n \in N$, $\{r_n(t)\}$ is non increasing $r_n(t) \to r(t)$ for some $0 \leq r(t) \leq 1$, and also $r_n(t) \leq$

$D(x_L, x_k, t)$ for all $L, k \geq n$.

Let ξ be the set of all continuous function

$F : [0,1]^3 \times [0,1] \to [-1,1]$

Such that F is nondecreasing on $[0,1]^3$ satisfying the following conditions .

(F_1) : $F((u, u, u), v) \leq 0$ implies that $v \geq y(u)$ where

$y : [0,1] \to [0,1]$ is a nondecreasing continuous function with

$y(s) > s$ for $s \in [0,1)$.

Example 4.7.5

$F : [0,1]^3 \times [0,1] \to [-1,1]$. defined the following :

1. $F((t_1, t_2, t_3), t_4) = y(\min\{t_1, t_2, t_3\}) - t_4$.

2. $F((t_1, t_2, t_3), t_4) = y(\sum_{i=1}^{3} a_i t_i) - t_4$ such that for every $a_i \geq 0$, $\sum_{i=1}^{3} a_i = 1$, where

$Y(s) = s^h$ for $0 < h < 1$.

In this section , our main result is the following theorem .

Theorem 4.7.6

Let (X , D , ±) be a complete bounded fuzzy metric space and T a self map of X satisfying for all $x, y \in X$ the implicit relation .

1 . $F(D(x, y, t), D(T_x, x, t), D(T_x, y, t), D(T_x, T_y, t)) \leq 0$

where $F \in \xi$.

Then , T has a unique fixed point p in X , and T is continuous at p .

Proof :

Let $x_0 \in X$ and $T_{X_n} = x_{n+1}$, let $r_n(t) = ₵_D = (A_n, t)$, where $A_n = \{x_n, x_{n+1}, \ldots\}$. then , we know $\lim_{n\to\infty} r_n(t) = r(t)$ for some $0 \leq r(t) \leq 1$. if $x_{n+1} = x_n$ for some $n \in N$, then T has a fixed point , $p \in X$, assume that $x_{n+1} \neq x_n$ for each $n \in N$. let $k \in N$ be fixed . taking $x = x_{n-1}$, $y = x_{n+m-1}$ in 1 , where $n \geq k$ and $m \in N$, we have :

$F(D(x_{n-1}, x_{n+m-1}, t), D(T_{X_{n-1}}, x_{n-1}, t), D(T_{X_{n-1}}, x_{n+m-1}, t), D(T_{X_{n-1}}, T_{X_{n+m-1}}, t))$

$= F(D(x_{n-1}, x_{n+m-1}, t), D(x_n, x_{n-1}, t), D(x_n, x_{n+m-1}, t), D(x_n, x_{n+m}, t)) \leq 0$

Thus , we have

$F(r_{n-1}(t), r_{n-1}(t), r_n(t), D(x_n, x_{n+m}, t)) \leq 0$

Since F is nondecreasing on $[0, 1]^3$. also , since $r_n(t)$ is nonincreasing we have:

$F(r_{k-1}(t), r_{k-1}(t), r_{k-1}(t), D(x_n, x_{n+m}, t)) \leq 0$

Which implies that :

$D(x_n, x_{n+m}, t) \geq y(r_{n-1}(t))$

Thus , for all n $\geq$ k , we have $\inf_{n \geq k}$ { D (x_n , x_{n+m} , t) } = $r_k(t) \geq y(r_{n-1}(t))$

Letting $k \to \infty$, we get r(t) $\geq$ y(r(t)) . if r(t) $\neq$ 1 , then :

r(t) $\geq$ y(r(t)) $>$ r(t) , which is a contradiction . thus r(t) = 1 , and hence $\lim_{n\to\infty} y_n(t) = 1$, thus , given ε > 0 , there exists an n_0 ϵ N such that $r_n(t) > 1 -$ ε .

Then , we have for n $\geq$ n_0 and m ϵ N , D (x_n , x_{n+m} , t) > 1 – ε . therefore , { X_n } is a Cauchy sequence in X . by the completeness of X , there exists a p ϵ X such that

$\lim_{n\to\infty} T_{Xn} = \lim_{n\to\infty} X_{n+1} = p$.

Taking x = x_n , y = p in 1 , we have :

F (D (x_n , p , t) , D (T_{Xn} , p , t) , D (T_{Xn} , x_n , t) , D (T_{Xn} , T_p , t))

= F (D (x_n , p , t) , D (x_{n+1} , p , t) , D (x_{n+1} , x_n , t) ,

D (x_{n+1} , T_p , t)) $\leq$ 0 .

Taking $\lim_{n\to\infty}$, we have :

F (D (p , p , t) , D (p , p , t) , D (p ,p , t) , D (p , T_p , t)) $\leq$ 0

Which implies that D (p , T_p , t) $\geq$ y (D (p , p , t)) = y (1) = 1

Hence , T_p = p . for the uniqueness , let p and w be fixed points of T . taking x = p , y = w in 1 , we have :

$F(D(p, w, t), D(T_p, p, t), D(T_p, w, t), D(T_p, T_w, t))$

$= F(D(p, w, t), D(p, p, t), D(p, w, t), D(p, w, t)) \leq 0$

Since F is nondecreasing on $[0, 1]^3$, we have :

$F(D(p, w, t), D(p, w, t), D(p, w, t), D(p, w, t)) \leq 0$

Which implies that $D(p, w, t) \geq y(D(p, w, t)) > D(p, w, t)$, which is a contradiction . thus , we have p = w . now , we show that T is continuous at p . let $\{y_n\}$ be a sequence in X and $\lim_{n\to\infty} y_n = p$. taking x = p , y = y_n in 1 , we have :

$F(D(P, Y_n, t), D(T_p, p, t), D(T_p, y_n, t), D(T_p, T_{Yn}, t))$

$= F(D(p, y_n, t), D(p, p, t), D(p, y_n, t), D(p, T_{Yn}, t)) \leq 0$

Which implies that $y(D(p, T_{Yn}, t)) \geq y(D(p, y_n, t))$. taking limit inf, we have :

$\lim_n \inf\{ y(D(p, T_{Yn}, t))\} \geq \lim_n \inf y(D(p, y_n, t)) =$

$y(1) = 1$.

Thus $\lim_{n\to\infty} T_{Yn} = p = T_p$. hence , T is continuous at p .

Theorem 4.7.7

Let (x , D , ±) be a compact fuzzy metric space and T a continuous self map of X satisfying for all x , y ϵ X with :

1 . $F (D (x , y , t) , D (T_x , x , t) , D (T_x , y , t) , D (T_x , T_y , t)) < 0$, where $F \in \xi$,

Then , T has a unique fixed point p in X .

Proof :

We know that for n = 1 , 2 , ….. , $T^n X$ is compact and $T^{n+1}X \subset T^nX$.

let $X_0 = \bigcap_{n=1}^{\infty} T^nX$. then , X_0 is a nonempty compact subset of X . and $TX_0 = X_0$, we claim that X_0 is a singleton set . suppose X_0 is not singleton . then , we know that the function :

$D : X \times X \times (0 , \infty) \to [-1 , 1]$ has a minimum value .

that is , there exists a $(x_0 , y_0 , t) \in X_0 \times X_0 \times (0 , \infty)$ such that

$D (x_0 , y_0 , t) \leq D (x , y , t)$ for all $x, y \in X_0$, since $TX_0 = X_0$, there exist $x_1 , y_1 \in X_0$ such that $TX_1 = X_0$, $T_{Y1} = y_0$. thus we have :

$F (D (x_1 , y_1 ,t) , D (x_0 , x_1 , t) , D (x_0 , y_1 , t) , D (x_0 , y_0 , t)) < 0$

And so $F (D (x_0 , y_0 , t) , D (x_0 , y_0 , t) , D (x_0 , y_0 , t) ,$

$D (x_0 , y_0 , t)) < 0$

Which implies that $D (x_0 , y_0 , t) > D (x_0 , y_0 , t)$. this is a contradiction. thus , x_0 is singleton , and hence T has a fixed point in X. uniqueness of fixed point of T follows from 1 .

Definition 4.7.8

We define a function $\Phi : [\,0 , 1\,] \rightarrow [\,0 , 1\,]$ s.t :

(p_1) Φ is strictly decreasing and left continuous .

(p_2) $\Phi($ ª $) = 0$ if and only if ª $= 1$

And we have $\lim_{ª \rightarrow 1-} \Phi($ ª $) = \Phi(1) = 0$

Theorem 4.7.9

Let (x , D , ±) be an D – complete fuzzy metric space and $T : X \rightarrow X$ is a self – map of X , and suppose that

$\Phi : [\,0 , 1\,] \rightarrow [\,0 , 1\,]$ s.t :

1. Φ is strictly decreasing and left continuous

2. $\Phi($ ª $) = 0$ if and only if ª $= 1$. let $k : (0 , ω) \rightarrow (0 , 1)$ be a function . if for any $t > 0$, T satisfies the following condition

$$\Phi(D(T_x , T_Y , t)) \leq k(t) . \Phi(D(x , y , t)) \qquad (1)$$

Where $x , y \in X$ and $x \neq y$, then T has a unique fixed point .

Proof :

Let x_0 be a point in X , define $x_{n+1} = T_{Xn}$ and $\tau_n(t) = D(x_n , x_{n+1} , t)$ for all $n \in N \cup \{0\}$, $t > 0$. now we first prove that T has a fixed point . the proof is divided into two cases .

Case 1:

If there exists $n_0 \in N \cup \{ 0\}$ such that $x_{n0+1} = x_{n0}$, i.e $T_{Xn0} = x_{n0}$, then it follows that X_{n0} is a fixed point of T .

Case 2 :

We assume that $0 < \tau_n(t) < 1$ for each n . that is to say , the relationship $x_n \neq x_{n+1}$ holds true for each n . from (1) , for every $t > 0$, we can obtain :

$$\Phi(\tau_n(t)) = \Phi (D(x_n , x_{n+1} , t))$$

$$= \Phi (D(T_{Xn-1} , T_{Xn} , t)) \leq k(t) . \Phi(\tau_{n-1}(t)) < \Phi (\tau_{n-1}(t)) \quad (2)$$

Since Φ is strictly decreasing , it is easy to show that $\{ \tau_n(t) \}$ is an increasing sequence for every $t > 0$ with respect to n .

we put $\lim_{n\to\infty} \tau_n (t) = \tau(t)$ and suppose that $0 < \tau(t) < 1$. by (2)

then $\tau_n(t) \leq \tau(t)$ implies that :

$$\Phi (\tau_{n+1}(t)) \leq k(t) . \Phi (\tau_n(t)) . (3)$$

For every t , by supposing that $n \to \infty$, since Φ is left continuous , we have:

$$\Phi(\tau(t)) \leq k(t) . \Phi(\tau(t)) < \Phi (\tau(t)) . (4)$$

Which is a contradiction , hence $\tau(t) \equiv 1$, that is , the sequence $\{\tau_n(t)\}$ converges to 1 for any $t > 0$.

next , we show that the sequence $\{ X_n \}$ is an D – Cauchy sequence , suppose that it is not . then there exist $0 < \varepsilon < 1$ and two sequences

$\{ p(n) \}$ and $\{ q(n) \}$ such that for every $n \in N \cup \{0\}$ and $t > 0$, we obtain that :

$p(n) > q(n) \geq n$, $D (x_{p(n)} , x_{q(n)} , t) \leq 1 - \varepsilon$.

$D(x_{p(n)-1} , x_{q(n)-1} , t) > 1 - \varepsilon$, and $D (x_{p(n)} , x_{q(n)} , t) > 1 - \varepsilon$ (5)

For each $n \in N \cup \{0\}$, we suppose that $s_n(t) = D(x_{p(n)} , x_{q(n)} , t)$, then we have $1 - \varepsilon \geq s_n(t) = D (x_{p(n)} , x_{q(n)} , t) \geq D (x_{p(n)-1} , x_{p(n)} , t/2) * D (x_{p(n)-1} , x_{q(n)} , t/2) > T_{p(n)} (t/2) * (1 - \varepsilon)$. (6)

Since $\tau_n(t/2) \to 1$ as $n \to \infty$ for every t , supposing that $n \to \infty$, we note that :

$\{ s_n(t) \}$ converges to $1 - \varepsilon$ for any $t > 0$, moreover by (1) , we have :

$\Phi (D (x_{p(n)} , x_{q(n)} , t)) \leq k(t) . \Phi (D (x_{p(n)-1} , x_{q(n)-1} , t)) < \Phi (D (x_{p(n)-1} , x_{q(n)-1} , t))$ (7)

According to the monotonicity of Φ .

we know that $D(x_{p(n)} , x_{q(n)} , t) > D(x_{p(n)-1} , x_{q(n)-1} , t)$ for each n . thus , on the basis of the formula (5) we can obtain :

$1 - \varepsilon > D(x_{p(n)} , x_{q(n)} , t) > D(x_{p(n)-1} , x_{q(n)-1} , t) > 1 - \varepsilon$ (8)

Clearly , this leads to a contradiction .

In particular , we consider a nother case .

that is , there exists $n_0 \in N \cup \{0\}$ such that $D(x_m, x_n, t) \leq 1 - \varepsilon$ for all m , $n \geq n_0$ .obviously , for any $p \in N$, we know that $D(x_{n0+p+2}, x_{n0+p+1}, t) \leq 1 - \varepsilon$.

as Φ is monotonic , it is easy to see that the sequence

$\{ D(x_{n0+p+2}, x_{n0+P+1}, t) = \sim$

for all $t > 0$. thus , we can obtain :

$$\Phi(D(x_{n0+p+2}, x_{n0+p+1}, t)) \leq k(t) . \Phi(D(x_{n0+p+1}, x_{n0+p}, t)) . \quad (9)$$

By supposing that $p \to \infty$, we have $\Phi(\sim) \leq 0$, which is also a contradiction .

Hence $\{ X_n \}$ is an D – cauchy sequence in the D – complete fuzzy metric space X .

there for , we conclude that there exists a point $x \in X$ such that:

$$\lim_{n \to \infty} x_n = x$$

now , we will show that x is a fixed point of T . since $0 < \tau_n(t) < 1$, there exists a subsequence $\{ X_{r(n)} \}$ of $\{ X_n \}$ such that $X_{r(n)} \neq X$ for every $n \in N$. from (2) , we can obtain :

$$0 \leq \Phi(D(x_{r(n)+1}, T_x, t)) = \Phi(D(T_{Xr(n)}, T_X, t)) \leq$$

$$k(t) . \Phi(D(x_{r(n)}, x, t)) \quad (10)$$

By supposing that $n \to \infty$ in (10) , we have :

$0 \leq \Phi(D (x , T_x , t)) \leq k(t) . \Phi (D (x , x , t)) = k(t) . \Phi(1) = 0$ (11)

So we can get $\Phi (D (x , T_x , t)) = 0$. according to the property (p_2)

It is easy to show that $D(x , T_x , t) = 1$, i.e , $T_x = x$.

We claim that X is the unique fixed point of T . assume that $y \neq x$ is a nother fixed point of T , we then obtain :

$\Phi (D (x , y , t)) = \Phi (D (T_x , T_y , t)) \leq k(t) . \Phi(D(x , y , t)) <$

$\Phi(D (x , y , t)$ (12)

Which is a contradiction . the proof of the theorem is now completed .

Example 4.7.10

Let X be the subset of R^2 defined by :

X = { A , B , C , D , E } , where A = (0, 0) , B = (1 , 0) ,

C = (1 , 2) , D = (0 , 1) , E = (1 , 3)

$\Phi (\tau) = 1 - \sqrt{\tau}$ for all $\tau \in [0 , 1]$ and $D (x , y , t) = e^{\frac{-2\, d(x ,y)}{t}}$, for all t > 0 , where d (x , y) denotes the Euclidean of R^2 .

clearly (x , D , *) is an D – complete fuzzy metric space with respect to the t – norm : a * b = a b

let $T : X \to X$ be given by :

$T(A) = T(B) = T(C) = T(D) = A$, $T(E) = B$

Define function $k : (0, \infty) \to (0, 1)$ as

$$K(t) = \begin{cases} 1 - e^{\frac{-4}{t}} & , 0 < t \leq 2 \\ \frac{t}{t+1} & , \quad t > 2 \end{cases}$$

One can see that the function Φ satisfies (p_1) and (p_2) , and the function k also satisfies the formula (1) , now , all the hypotheses of theorem 4.7.3 are satisfied and thus T has a unique fixed point , that is X = A .

Theorem 4.7.11

Let (x , D , ±) be a compact fuzzy metric space and T a continuous self – map of X and suppose that $\Phi : [0, 1] \to [0, 1]$ satisfies the foregoing (p_1) and (p_2) . if for any $t > 0$, T satisfies the following condition :

$$\Phi (D (T_X , T_y , t)) < \Phi (x , y , t)) \qquad (*)$$

Where $x , y \in X$ and $x \neq y$. thus T has a unique fixed point .

Proof :

Let x_0 be a point in X . define $X_{n+1} = T_{Xn}$, with $n \in N \cup \{ 0 \}$, and the sequence $\{ T^n x_0 \}$. if $T^n x_0 = T^{n+1} x_0$ for some n , then there exists $z \in X$ such that $z = T^k x_0$ for all $k \geq n$. so we assume that $T^n x_0 \neq T^{n+1} x_0$ for every n . since (x , D , ±) is a compact fuzzy metric space , there exists a subsequence $\{ T^{K(n)}x_0 \}$ of $\{ T^n x_0 \}$ such that :

$T^{K(n)}x_0 \rightarrow x \in X$ as $n \rightarrow \infty$. according to the continuity of T , we have

$\lim_{n\to\infty} X_{K(n)+1} = T(x)$ and $\lim_{n\to\infty} X_{K(n)+2} = T^2(x)$

As Φ is left continuous , it follows that :

$$\Phi (D (x , T_x , t)) = \lim_{n\to\infty} \Phi (D (X_{K(n)} , X_{K(n)+1} , t))$$

$$= \lim_{n\to\infty} \Phi (D (X_{K(n)+1} , X_{K(n)+2} , t)) = \Phi (D (T_x , T^2{}_X , t)) \quad (**)$$

Now , we show that $T_X = x$. otherwise , by (*) we can obtain :

$$\Phi (D (T_x , T^2{}_x , t)) < \Phi (D (x , T_x , t)) .$$

Obviously , this is contrary to the formula (**) .

The proof of uniqueness is similar to that of theorem 4.7. 3 .

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جامعة النجاح الوطنية
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قدمت هذه الأطروحة استكمالاً لمتطلبات الحصول على درجة الماجستير في الرياضيات بكلية الدراسات العليا في جامعة النجاح الوطنية في نابلس، فلسطين.

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الملخص

في هذه الرسالة قمنا بالتحري عن الفضاءات القياسية الضبابية باستخدام تعريفات ووجهات نظر مختلفة بعضها تم تطبيقه على المجموعات العادية والبعض الآخر تم تطبيقه على مجموعة النقاط الضبابية.

لقد تم كذلك دراسة مفهوم الاكتمال للفضاءات القياسية الضبابية باستخدام الثوابت الضبابية وايجاد النتائج في البيئة الضبابية والتي توازي مثيلاتها في التحليل الكلاسيكي.

وأخيراً, تم اثبات عدد من نظريات النقاط الثابتة الضبابية للفضاءات القياسية الضبابية.