An-Najah National University
Faculty of Graduate Studies

Extending Topological Properties to Fuzzy Topological Spaces

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الإقرار

أنا الموقعة أدناه مقدمة الرسالة التي تحمل العنوان:

Extending Topological Properties to Fuzzy Topological Spaces

توسعة الخصائص التبولوجية للفراغات التبولوجية الضبابية

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Declaration

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

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Extending Topological Properties to Fuzzy Topological Spaces

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Abstract

In this thesis the topological properties of fuzzy topological spaces were investigated and have been associated with their duals in classical topological spaces.

Fuzzy sets, fuzzy functions and fuzzy relations were presented along with their properties. Different types of fuzzy topological spaces (FTS) were introduced in Chang's and Lowen's sense as well as intuitionistic (FTS). Many topological properties were proved to be extensions to those in non fuzzy setting, while examples were presented for those non extension properties. For instance, the closure of the product is not equal to the product of the closures.

Also different approaches of separation axioms were investigated using Q-neighborhoods and fuzzy points, it turns out that most of them are not extension of classical separation axioms.

Fuzzy topological properties are considered, for instance, we studied fuzzy connectedness and fuzzy compactness. It is found that the product of an infinite number of fuzzy compact spaces may not be compact.

Finally, fuzzy continuity, fuzzy almost continuity and fuzzy $\boldsymbol{\delta}$-continuity were introduced with a theorem proved the way they are related.

Introduction

The concept of fuzzy sets was first introduced by Lotfi Zadeh in1965 [38], then later on; Chang in [6] introduced the concept of fuzzy topological space as an extension to classical topological space. After that, many authors studied the topological properties under fuzzy settings.They suggested different definitions for the same property which lead to different approaches. In this thesis we study and investigate many of those topological properties. We found that there were a lot of agreement between properties in fuzzy and nonfuzzy setting, but also there were a lot of differences.

In chapter one we concentrate on the concept of fuzzy sets and their behaviors through set operations, which act in similar manner with regular sets. Also, we went through some special types of fuzzy sets called fuzzy points and fuzzy singletons, and explore the different relations which relate them to fuzzy sets.We also extend any function f between any two regular sets; f:X $\rightarrow$ Y; to a fuzzy function $\bar{f}$ between two families of fuzzy subsets; $\bar{f}$:F(X) $\rightarrow$ F(Y). Finally we show that; among other properties; quasi coincident relations are preserved under fuzzy functions and show the relationship between the product of fuzzy functions and the fuzzy function of the product space.

In chapter two, we started with the definition of a fuzzy topological space as an extension to classical topological space in both Chang's view [6] and Lowen's view [17]. We present the notion of fuzzy interior and fuzzy closure with their properties. After that we classify different types of membership between fuzzy points and fuzzy sets which affects some properties of “belonging”. Followed by definitions of fuzzy neighborhood and Q-neighborhood systems, also fuzzy bases and fuzzy subbases were presented. Then, the product fuzzy topology is introduced as well as its properties concerns the fuzzy closure and fuzzy interior. Finally, we present a generalization of fuzzy sets, namely, intuitionistic fuzzy sets and the fuzzy topology they generalize, namely intuitionistic fuzzy topological spaces which have been greatly studied by many authors.

In chapter three we study the extension of the separation axioms to a fuzzy setting. We started with fuzzy Hausdorffness presenting three different approaches. The first one is using fuzzy points and fuzzy neighborhoods, while the second uses fuzzy points and Q-neighborhoods, and the third approach uses crisp points of the set X. We then show that these three different approaches are equivalent.

Another type of fuzzy Hausdorffness is using the α-level(α-Hausdorffness) and then it is concluded that the space is Hausdorff if and only if it is α-Hausdorff for each $\alpha \in [0,1]$. After that we went through other separation axioms defining T_0, T_1, T_2 and $T_{2\frac{1}{2}}$ using fuzzy points and fuzzy neighborhoods,under those definitions it is proved that a fuzzy topological space is T_1 if and only if every crisp singleton is closed, which is not the exact property of T_1 spaces in the nonfuzzy setting. This suggested a stronger definition of T_1 space (T_s space) where the statement "every fuzzy singleton is closed" is valid. After that fuzzy regular and normal spaces are defined, and some of their properties were presented. Finally; separation axioms in intuitionistic fuzzy topological spaces were defined and called IFT_1 and IFT_2,… etc.

Chapter four deals with the concept of fuzzy connectedness and fuzzy compactness. We have chosen a definition of fuzzy connectedness (where $\bar{0}$, $\bar{1}$ are the only fuzzy clopen subsets of X), other equivalent definitions of fuzzy connectedness were presented. Concerning the extension of the connectedness property from nonfuzzy setting to fuzzy setting, it is found that in fuzzy setting the product of fuzzy connected spaces may not be fuzzy connected,contrary to the property in nonfuzzy setting. Some other characterizations of fuzzy connectedness were presented and proved. Then fuzzy compactness is defined using fuzzy open cover and the finite intersection property parallel to compactness in regular space. Some properties were extended. For instance, it is shown that "every fuzzy closed subset of a fuzzy compact space is fuzzy compact", " the fuzzy continuous image of a fuzzy compact space is fuzzy compact" and " the finite product of fuzzy compact spaces is fuzzy compact". This complies with the classical topological spaces. But, the product of an infinite number of fuzzy compact spaces may not be fuzzy compact contrary to the nonfuzzy setting.

In chapter five we studied fuzzy continuous functions and explore both local and global properties and prove that they are equivalent. After that new types of fuzzy continuity were presented namely “ almost continuity, $\boldsymbol{\delta}$-continuity “ where new types of fuzzy open sets called “fuzzy regular open sets” were used. Also “ fuzzy precontinuity “ using fuzzy preopen sets. Finally “Generalized fuzzy continuity” using generalized fuzzy sets.

Chapter One

Fuzzy Sets and Fuzzy Functions

Chapter One

Fuzzy Sets and Fuzzy Functions

Introduction

Fuzzy sets, in Mathematics, are sets having elements with a membership degree.This concept of sets was first generalized by Professor Lotfi A. Zadeh in 1965 in his famous paper [38] where the concept of fuzzy sets was introduced, it was specifically designed for representing uncertainty in mathematics and for dealing with vagueness in many real life problems, it is suitable for approximating reasoning mathematical Models that are hard to derive or giving a decision with incomplete information. In classical set theory, an element either belongs or doesn't belong to the set,it is not the case in fuzzy setting, here, it has a membership degree between zero and one,which describes the new definition of the characteristic function. In this chapter we will first give definitions of fuzzy sets,then we show some operations on them and properties involving these operations. Also we will introduce the concept of fuzzy points as a special case of fuzzy subsets, then we define fuzzy functions as an extension of functions between pairs of sets and explore the properties of fuzzy operations of fuzzy sets and fuzzy points on fuzzy functions.

1.1 Fuzzy Sets and Fuzzy Operations

In set theory a subset A of a set X can be identified with the Characteristic function χ_A that maps X to {0,1} in a way where all elements of A go to 1, while X-A elements go to 0.

i.e $\chi_A(x) = \begin{cases} 1 & if\ x \in A \\ 0 & if\ x \in X - A \end{cases}$

and therefore, there is a natural 1-1 correspondence between the family of all subsets of X and the family of the characteristic functions on X.

Zadeh in [38] extended the definition of the characteristic functions by replacing the set {0,1} by the closed interval [0,1] which is the bases to the new definition of fuzzy sets.

Definition 1.1.1:

Let X be a regular set, a fuzzy subset of X is a function μ_A that maps X to the closed interval [0,1]. In other words, $\mu_{A:}$ X → [0,1] and $\mu_A(x)$ is called the grade of membership of the element x.

In the case of the characteristic function $\chi_{A:}X \rightarrow \{0,1\}$ if $\chi_A(x) = 0$ then; the grade of membership is 0; and this means that x doesn’t belong to A, if the characteristic function $\chi_A(x) = 1$,then the grade of membership is 1;and this means that x belongs to A. But, in the case of fuzzy sets: μ_A (x) could be any other number from 0 to 1.

Example 1.1.2:

$\mu_A(x) = 0.9$ may mean that x is more likely to be in $\mu_{A,}$or if $\mu_A(x)=0.5$ then x may be half way between belonging to μ_A and not belonging to μ_A. It is clear that regular subsets of X are a special case of fuzzy sets called crisp fuzzy sets where $\mu_A(x) \in \{0,1\} \leq [0,1]$.

We use different ways to represent a fuzzy subset of X.In the following example we describe some of those ways:

Example 1.1.3:

Consider the regular set X where X={a,b,c,d,e} and let μ_A be the fuzzy subset of X that maps X to [0,1] by mapping:

a→0.1, b→0.8, c→0.5,d→0, and e→0.4.

We may represent μ_A as the set of ordered pairs:

μ_A = {(a, 0.1),(b, 0.8),(c,0.5), (d,0),(e, 0.4)} using regular set notation, or we may write it as $\mu_A = \{a_{0.1,} b_{0.8,} c_{0.5}, d_0, e_{0.4}\}$.This last form will be mostly used in this manuscript.

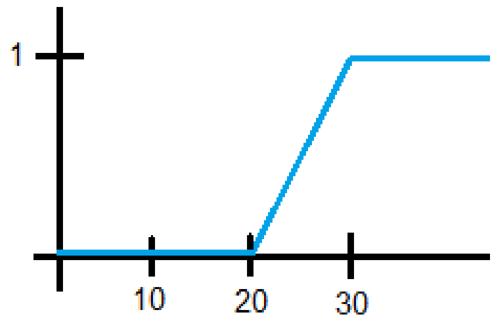
Another example that explains the concept of the grade of membership is the following:

Example 1.1.4:

Take X to be a set of people, a fuzzy subset OLD may be defined to be the answer of the question "to what degree a person x is old ? "the answer could come in a membership function based on a person's age

$$\text{OLD}(x) = \begin{cases} o & if\ x < 20 \\ \frac{age\ of\ x - 20}{10} & if\ 20 \le x < 30 \\ 1 & if\ x \ge 30 \end{cases}$$

Graphically we have:



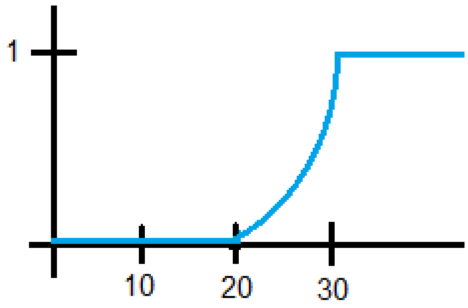
We may say that the percentage of belonging for any person with age > 30 to being OLD is 100%, while a person with age 29 years old has a percentage of 90% and we write:

OLD(29) = 0.9 or 90% and OLD(25)=0.5 or 50%

This grade of membership function is linear. But we may have the nonlinear function that reflects the importance of the age needed. For example:

$$\text{OLD(x)} = \begin{cases} o & if\ x < 20 \\ \frac{1}{100}(age\ of\ x - 20)^2 & if\ 20\ \leq x < 30 \\ 1 & if\ x \geq 30 \end{cases}$$

And graphically it is:

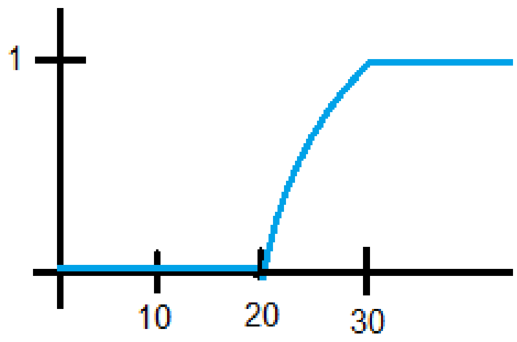


Now,OLD(29) =.81 or 81%,OLD (25)= 0.25 or 25% which is less than 50% (in the linear case) so being less than and away from 30 looses more importance than in the linear case.

On the other hand we may have the function

$$\text{OLD(x)} = \begin{cases} o & if\ x < 20 \\ 1 - \frac{1}{100}(age\ of\ x - 30)^2 & if\ 20\ \leq x < 30 \\ 1 & if\ x \geq 30 \end{cases}$$

Which has the graph:



In this case:OLD(29) =.99 or 99% and OLD (25)= 0.75 or 75% which is more than 50%(linear case). This membership grade function reflects that being close to,but less that 30 gains more importance than in the linear case.

There are other types of fuzzy subsets, the fuzzy constant subset of X is one which is the function that takes all elements of X to a constant c, where c ϵ [0,1], and it is denoted by $\bar{c}$.

Special fuzzy constant subsets are $\bar{1}$ and $\bar{0}$, where,

$\bar{1}$: is the fuzzy subset of X that takes all the elements of X to 1 and $\overline{0}$ is the fuzzy subset of X that takes all the elements of X to 0.

1.2 Operations on Fuzzy Sets

After these new concepts of fuzzy sets were defined, suitable operations on them should be performed that extend the usual operations on sets including the union, intersection and complementation as follows:

Definition 1.2.1 [35]:

Let μ_A and μ_B be two fuzzy subsets of X, $\mu_A \Lambda\ \mu_B$, $\mu_A V\ \mu_B$, $\mu_A{}^c$ are fuzzy subsets of X defined as follows:

$(\mu_A\ \Lambda\ \mu_B)(x) = \min\{\ \mu_A\ (x), \mu_B\ (x)\}$.

$(\mu_A\ V\ \mu_B)(x) = \max\{\ \mu_A\ (x), \mu_B\ (x)\}$.

$\mu_A{}^c(x) = 1- \mu_A\ (x)$.

These definitions are generalized to any number of fuzzy subsets of X,so; for any family $\{\ \mu_{A\,\alpha}: \alpha \in \Delta\}$ of fuzzy subsets of X, where Δ is an indexing set, we define:

$(V_\alpha\ \mu_{A\alpha})(x) = \sup\ \{\ \mu_{A\alpha}(x)\ \alpha \in \Delta\}$

$(\Lambda_\alpha\ \mu_{A\alpha})(x) = \inf\ \{\ \mu_{A\alpha}\ (x): \alpha \in \Delta\}$

We illustrate the previous definitions by the following examples.

Example 1.2.2:

(1) Take the fuzzy subsets

$\mu_A = \{a_{0.3}, b_{0.8}, c_0, d_{0.98}\}$ and $\mu_B = \{a_{0.8}, b_{0.1}, c_{0.1}, d_{0.3}\}$

then: $\mu_A \wedge \mu_B = \{a_{0.3}, b_{0.1}, c_0, d_{0.3}\}$

$\mu_A \vee \mu_B = \{a_{0.8}, b_{0.8}, c_{0.1}, d_{0.98}\}$ and $\mu_A{}^c = \{a_{0.7}, b_{0.2}, c_1, d_{0.02}\}$

(2) Take an infinite number of fuzzy subsets.

Let X={a,b},

$\mu_{A1}=\{ a_{0.49}, b_{.21} \}$

$\mu_{A2}=\{ a_{0.499}, b_{.201} \}$

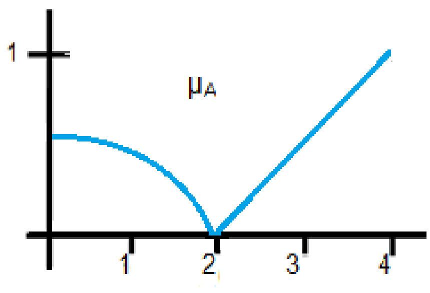
$\mu_{A3}=\{ a_{0.4999}, b_{.2001} \}$

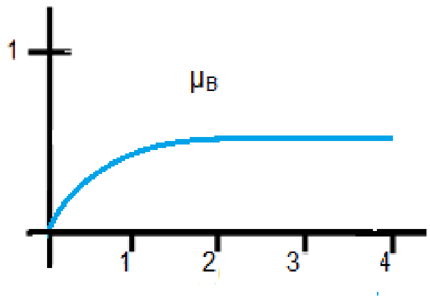
⋮

Then $\bigvee_{i=1}^{\infty} \mu_{A_i} =\{ a_{0.5}, b_{.21} \}$ and $\bigwedge_{i=1}^{\infty} \mu_{A_i} =\{ a_{0.49}, b_{.2} \}$

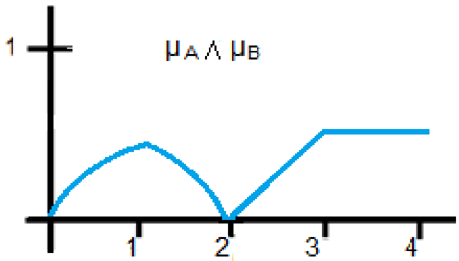
(3) For the continuous graph case:

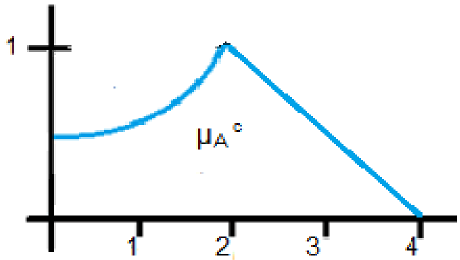
Take X=[0,4], μ_A and μ_B are as follows:





Then $\mu_A \Lambda \mu_B$, and $\mu_A{}^c$ are as follows:





To show that this definition extends the union, intersection and complementation applied on regular subsets of X, we have:

$(\mu_{AVB})(x) = \max\{ \mu_A (x), \mu_B (x)\}$.

In case, x ϵA or x ϵB then $\mu_A(x)=1$ or $\mu_B (x)=1$ which implies that $\max \{ \mu_A (x), \mu_B (x) \} = 1$ so $(\mu_{AVB})(x)=1$ i.e. x ϵAUB

But, if x∉A and x∉B then $\mu_A (x)=0$ and $\mu_B (x)=0$ Which implies that $\max \{ \mu_A (x), \mu_B (x) \} =0$ and $(\mu_{AVB})(x)=0$

so x ∉ AUB, which complies with the regular definition of "union".

In similar manner, we may show the same for intersection and complementation

We will see in the next theorem that we can extend Demorgan's Laws from regular (crisp) sets to fuzzy subsets:

Theorem: 1.2.3 [35]

Let μ_A and μ_B be two fuzzy subsets of X, we have:

1. $(\mu_A \Lambda \mu_B)^c (x)= (\mu_A{}^c V \mu_B{}^c) (x)$.

2. $(\mu_A \vee \mu_B)^c (x) = (\mu_A{}^c \wedge \mu_B{}^c)(x)$.

Proof:

1) $(\mu_A \wedge \mu_B)^c (x) = 1 - \min\{\mu_A(x), \mu_B(x)\}$

$$= \begin{cases} 1 - \mu_A(x) & \text{if } \mu_A(x) \leq \mu_B(x) \\ 1 - \mu_B(x) & \text{if } \mu_B(x) \leq \mu_A(x) \end{cases}$$

$$= \begin{cases} 1 - \mu_A(x) & \text{if } 1 - \mu_A(x) > 1 - \mu_B(x) \\ 1 - \mu_B(x) & \text{if } 1 - \mu_B(x) > 1 - \mu_A(x) \end{cases}$$

$= \max\{1 - \mu_A(x), 1 - \mu_B(x)\}$

$= \max\{\mu_A{}^c(x), \mu_B{}^c(x)\}$

$= (\mu_A{}^c \vee \mu_B{}^c)(x)$

2) $(\mu_A \vee \mu_B)^c (x) = 1 - \max\{\mu_A(x), \mu_B(x)\}$

$$= \begin{cases} 1 - \mu_A(x) & \text{if } \mu_A(x) \geq \mu_B(x) \\ 1 - \mu_B(x) & \text{if } \mu_B(x) \geq \mu_A(x) \end{cases}$$

$$= \begin{cases} 1 - \mu_A(x) & \text{if } 1 - \mu_A(x) < 1 - \mu_B(x) \\ 1 - \mu_B(x) & \text{if } 1 - \mu_B(x) < 1 - \mu_A(x) \end{cases}$$

$= \min\{1 - \mu_A(x), 1 - \mu_B(x)\}$

$= \min\{\mu_A{}^c(x), \mu_B{}^c(x)\}$

$= (\mu_A{}^c \wedge \mu_B{}^c)(x)$

This theorem can be generalized to any family of fuzzy subsets of X. specifically:

$(\vee_\alpha \mu_{A_\alpha})^c = (\wedge_\alpha \mu_{A_\alpha^c})$ and $(\wedge_\alpha \mu_{A_\alpha})^c = (\vee_\alpha \mu_{A_\alpha^c})$

Now we compare two fuzzy subsets of a set X as one of them containing the other as follows:

Definition 1.2.4 [35]:

Let μ_A, μ_B be two fuzzy subsets of X, we say $\mu_A \leq \mu_B$ to mean $\mu_A(x) \leq \mu_B(x)$ for all x∊ X.

For example:

Consider X = {a, b, c, d }, and let $\mu_B = \{a_{0.4}, b_{0.8}, c_{0.1}, d_0\}$ and

$\mu_A = \{a_{0.1}, b_{0.8}, c_0, d_0\}$, then clearly $\mu_A \leq \mu_B$

one of the basic notions of fuzzy sets is the notion of α-level

Definition:1.2.5 [35]:

The α-level of μ_A denoted by μ_A^{α} is a subset of X, where the grade of membership of its elements ≥ α. That is, $\mu_A^{\alpha} = \{x \in X: \mu_A(x) \geq \alpha\}$, where $\alpha > 0$

We define the 0-level in case of X is the real line by

μ_A^{0} =the closure of($\{x \in X: \mu_A(x) > 0\}$) in R.

The support of μ_A is defined as the set of all elements of X with nonzero membership and is denoted by supp of μ_A that is,

$\text{supp}(\mu_A) = \{x \in X: \mu_A(x) > 0\}$.

The following example displays some α-levels of some fuzzy subsets:

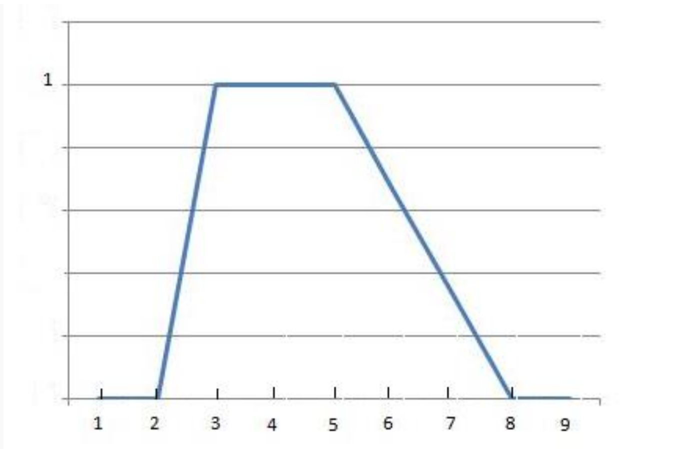
let μ_A = {$a_{0.4}$, $b_{0.7}$, $c_{0.3}$, $d_{0.2}$} be a fuzzy subset of X = {a,b, c,d } Then the 0.3-level = $\mu_A^{0.3}$= {a,b,c}, the 0.1 level = $\mu_A^{0.1}$={a,b,c,d}. And the support (μ_A) = X = {a, b, c, d }

We say that a fuzzy set μ_A in X, where X is infinite, is countable whenever supp(μ_A) is countable.

The following example computes some α-levels:

Example: 1.2.6

The following represents the graph of a fuzzy subset of R=(-∞,∞) with its function representation.



$$\text{where } \mu_A(x) = \begin{cases} x-2 & if\ x \in [2,3] \\ 1 & if\ x \in [3,5] \\ \frac{8-x}{3} & if\ x \in [5,8] \\ 0 & elsewhere \end{cases}$$

the 0.4 level of this fuzzy set is, $\mu_A^{0.4} = \{x \in X: \mu_A(x) \geq 0.4\}$

$0.4 \leq x-2 \Rightarrow x \geq 2.4$

$0.4 \geq \frac{8-x}{3} \Rightarrow x \leq 6.8$,so $\mu_A{}^{0.4}= [2.4, 6.8]$

In general,the α-level can be found as follows:

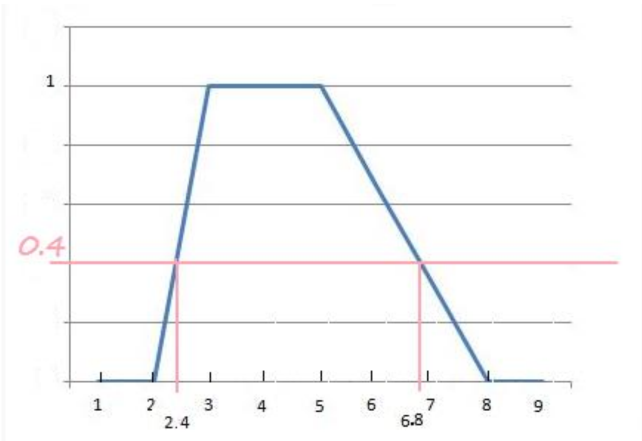
$\mu_A{}^{\alpha} = [x_1^{\alpha}, x_2^{\alpha}]$

Now, $\alpha = x_1^{\alpha} - 2$, and this implies that $x_1^{\alpha}=\alpha+2$

And $\alpha=\frac{8-x_2^{\alpha}}{3}$ which means $x_2^{\alpha}= 8-3\alpha$

So $\mu_A{}^{\alpha} = [\alpha+2, 8-3\alpha]$

For $\alpha= 0.4$, $\mu_A{}^{0.4}= [2.4, 6.8]$



1.3 Fuzzy Points and Fuzzy Singletons

As a special case of fuzzy subsets of X are the fuzzy points. They were defined by Wong [34], and later on, other definitions were presented by Srivastava [32] and Ming and Liu [22]

Definition 1.3.1:[34]:

Let X be a regular set, a fuzzy point p is a fuzzy subset of X that takes an element a to a number λ such that λ ϵ (0,1) and takes the remaining elements to zero, and it will be denoted by $p = a_\lambda$.

Support (p) = {a}, p (a)= λ, and p(X-{a})=0.

We define a fuzzy singleton x_r in X as a fuzzy subset in X which takes an element x to r where rϵ(0,1], and takes everything else to zero for example: if X={a, b, c, d}, then a fuzzy point $a_{0.3}$ is the fuzzy subset {$a_{0.3}$, b_0, c_0, d_0}

Remark:

From now on we will use A instead of μ_A as a notation for fuzzy subset, and we use F(X) to be the family of all fuzzy subsets of X.

Definition 1.3.2 [34]:

Let p be the fuzzy point a_λ, and A be a fuzzy subset of X, since a_λ is a fuzzy subset of X,we may define p ϵ A if and only if $\lambda \le A(a)$

That is a_λ ϵ A if and only if $\lambda \le A(a)$

For example:let X= {a, b, c, d} and A={$a_{0.4}$, $b_{0.5}$, $c_{0.3}$,$d_{0.9}$} then: $b_{0.1}$ ϵA

but $c_{0.9}$ ∉A.

Definition:1.3.3 [22]

A fuzzy singleton x_r in X is said to be quasi-coincident (in short Q-coincident) with a fuzzy set A in X if and only if $r + A(x) > 1$ and this is denoted by x_r Q A

Remark: it is clear that a_λ Q A $\Leftrightarrow a_\lambda \notin A^c$

Definition 1.3.4 [22]:

A fuzzy subset A in X is called Q-coincident with a fuzzy subset B in X (denoted by A Q B) if and only if $A(x) + B(x) > 1$ for some x in X

1.4 Fuzzy Functions

Now, we introduce the fuzzy function concept between two families of fuzzy subsets corresponding to a function between two crisp sets

Definition 1.4.1 [35]:

Let X and Y be two regular sets, and let f: X →Y be any function.

For any fuzzy subset A of X; we define:

$\bar{f}$: F(X) → F(Y), by $\bar{f}$(A) to be the fuzzy subset of Y defined by:

$$\bar{f}(A)(y) = \begin{cases} \text{Sup}\,\{A(x): x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi \end{cases}$$

and we define the fuzzy function $(\bar{f})^{-1}$ as $(\bar{f})^{-1}$ (B) for any fuzzy subset B of Y by:

$(\bar{f})^{-1}$ (B) (x) = B (f(x)).

Now, we consider examples that clarify the above definition

Example 1.4.2 (1):

Take X= {a, b, c, d}, Y= {u, v, w}

and f: X → Y by: a → u,b → v, c → v and d → v. Let A be the fuzzy subset of X such that A={$a_{0.2}$,$b_{0.5}$,$c_{0.6}$,d_0 }, then $\bar{f}$(A) is the fuzzy subset of Y defined as:

$\bar{f}$ (A): Y → [0,1]:

u → 0.2, v → max{0.5, 0.6,0} = 0.6,and w → 0

Example 1.4.2 (2)

Let X = {a, b, c, d}, Y= {u, v, w} and f: X → Y be the function that maps a to u and b, c and d to v, and let B: Y → [0,1] to be the fuzzy subset of Y that maps u to 0.3, v to 0.5, and w to 0.8.

Then $(f)^{-1}$ (B): X → [0,1]

a → 0.3, b → 0.5,c → 0.5 and d → 0.5

The following definition concerns the product of two fuzzy sets

Definition [35]:

Let A be a fuzzy subset of X and B be a fuzzy subset of Y. Then we consider AxB to be the fuzzy subset of XxY defined by:

(AxB) (x,y)= min{ A(x),B(y) } for (x,y)∈XxY

Following this definition we have the following remark:

Remark 1.4.3:

For any fuzzy subset A of X and fuzzy subset B of Y, we have: $(AxB)^c$ = (A^c x $\overline{1_y}$) V ($\overline{1_x}$ x B^c) where $\overline{1_x}$ is the fuzzy set that maps all element of X to 1 while $\overline{1_y}$ is the fuzzy set that maps all elements of Y to 1.

Proof:

(1 – AxB)(x,y) = max {1-A(x), 1-B(y) }

= max { (A^c x1)(x,y), (1x B^c)(x,y) }

= [(A^c x $\overline{1_y}$) V ($\overline{1_x}$x B^c)](x,y) for every (x,y)∈XxY

Also we can define the product of two fuzzy functions as follows:

Definition 1.4.4 [37]:

let f_1: $X_1 \to Y_1$, $\overline{f_1}$:F(X_1) → F(Y_1) and f_2: $X_2 \to Y_2$, $\overline{f_2}$:F(X_2) → F(Y_2)

Then: f_1 x f_2: X_1 x $X_2 \to Y_1$ x Y_2 is defined by:

(f_1 x f_2)(x_1,x_2) = (f_1(x_1), f_2(x_2)) for every (x_1,x_2)∈ X_1 x X_2.

And therefore for any fuzzy subsets A_1,A_2 of X_1 and X_2 respectively

$\overline{(f_1 \times f_2)}$ (A_1 x A_2)(y_1,y_2) =

$$\begin{cases} \sup\{(A_1 \times A_2)(x_1,x_2): (x_1,x_2) \in (f_1 \times f_2)^{-1}(y_1,y_2) & if\ (f_1 \times f_2)^{-1}(y_1,y_2) \neq \emptyset \\ 0 & if\ (f_1 \times f_2)^{-1}(y_1,y_2) = \emptyset \end{cases}$$

and: $(\overline{f_1 \times f_2})^{-1}$ $(B_1 x\ B_2)(x_1,x_2)$= $(B_1xB_2)((f_1xf_2)\ (x_1,x_2))$, for any fuzzy subsets B_1 and B_2 of Y_1 and Y_2 respectively.

Theorem: 1.4.5 [37]

Under the assumption of the previous definition, we have

$(\overline{f_1 \times f_2})^{-1} (B_1 x\ B_2) = (\overline{f_1})^{-1} (B_1)\ x\ (\overline{f_2})^{-1} (B_2)$

Proof:

for every $(x_1, x_2) \in X_1\ x\ X_2$ we have:

$(\overline{f_1 \times f_2})^{-1} (B_1 x\ B_2)\ (x_1, x_2) = (B_1 x\ B_2)(f_1(x_1), f_2(x_2))$

So $(\overline{f_1 \times f_2})^{-1} (B_1 x\ B_2)\ (x_1, x_2)$ = min $(B_1(f_1(x_1), B_2(f_2(x_2)))$

= min $(f_1^{-1}(B_1)(x_1), f_2^{-1}(B_2)(x_2))$

= $((\overline{f_1})^{-1} (B_1)\ x\ (\overline{f_2})^{-1} (B_2))\ (x_1, x_2)$

We cosider the fuzzy graph of a fuzzy function.

In regular setting, for any function f: X→Y we define the graph of f, G_f, to be the function g: X →XxY defined by: g(x) = (x, f(x)) for every x∈X. So G_f = { (x, f(x)): x∈X }

Definition: 1.4.6

Let f: X→Y be any function and $\overline{f}$:F(X) → F(Y) be the corresponding fuzzy function. The fuzzy graph of $\overline{f}$ is the fuzzy function $\overline{g}$ where, $\overline{g}$: F(X) → F(X) x F(Y) defined by:for a fny fuzzy subset A of X, $\overline{g}$(A) = AxB, for any fuzzy subset B of Y

Remark 1.4.7 [37]

Under fuzzy setting we have $(\overline{g})^{-1}$(AxB) = A ∧ $\overline{(f)}^{-1}$(B)

proof:

$(\overline{g})^{-1}$(AxB)(x) = A x B(g(x))

= A x B(x,f(x))

= min { A(x), B(f(x)) }

= A ∧ $\overline{f}^{-1}$ (B)(x)

The following theorem shows that the image of a fuzzy point in Xis a fuzzy point in Y, but the inverse image of a fuzzy point in Y may not be a fuzzy point in X

Theorem 1.4.8 [35]:

(1) If p = a_λ is a fuzzy point in X,with support *a*, and with

value =λ, then $\overline{f}$ (p) is a fuzzy point in Y, call it q,where

$\overline{f}$ (p) = f(a)$_\lambda$ = q such that f(a) is the support of q and λ is the

value of q

Proof:

If f^{-1}(y) = ø,q(y) =0

If $f^{-1}(y) \neq ø$,q(y) =sup { p(x): x∊ $f^{-1}(y)$ } here, there are two cases:

case one: if a ∊ $f^{-1}(y)$

$q(y) = \sup \{ p(x): x \in f^{-1}(f(a)) \} = \{ \lambda, 0, 0, \dots \} = \lambda$

case two a ∉ $f^{-1}(y)$

$q(y) = \sup \{ 0, 0, \dots \} = 0$

(2) If $q = b_r$ fuzzy point in Y then $f^{-1}(q)$ may not be fuzzy

point in X

The following examples explain this result.

Example (1):

Suppose $f^{-1}(b)$ is not a singleton,say $f^{-1}(b) = \{ \alpha, \beta \}$

then $f^{-1}(q) = \{ \alpha_r, \beta_r, 0, 0, \dots \}$ which is not a fuzzy point

Example (2):

If $f^{-1}(b) = ø$, then $f^{-1}(q) = ø$ which in not a fuzzy point. According to the previous two examples if $f^{-1}(b)$; where $q = b_r$; is a singleton then if $f^{-1}(q)$ is a fuzzy point in X.

The following theorem shows the effect of fuzzy functions on the quasi-coincident relation between a fuzzy point and a fuzzy set,

Theorem 1.4.9[37]:

Let $f:X\rightarrow Y$ be a function, then for any fuzzy point p= a_λ and for any fuzzy subset A of X, we have:if p Q A then f(p) Q f(A)

proof:

Let p = a_λ and f(p) = $f(a)_\lambda$

since p Q A then: $\lambda + A(a) > 1$

Consider $\lambda + f(A)(f(a))$

$\lambda + f(A)(f(a)) = \lambda + \sup \{ A(x) : x \in f^{-1}((f(a)) \}$

$\geq \lambda + A(a) > 1$

Theorem 1.4.10 [37]:

If $q=b_\lambda$ and $f^{-1}(b)$ is a singleton ($f^{-1}(b) = \{a\}$) then $f^{-1}(q)$ is the fuzzy point $=a_\lambda$ and in this case: if q Q B then $f^{-1}(q)$ Q $f^{-1}(B)$

proof:

We have qQB which means $\lambda + B(b) > 1$

Now

$\lambda + \overline{f}^{-1}(B)(a) = \lambda + B(f(a))$

$= \lambda + B(b)$

> 1

That is,$f^{-1}(q) Q f^{-1}(B)$

Chapter Two

Fuzzy Topological Spaces

Chapter Two
Fuzzy Topological Spaces

Introduction

General topology was one of the first branches of pure mathematics that have been applied to the fuzzy settings. After three years of the introduction of fuzzy sets by Zadeh in 1965,Chang, in [6] in 1968,gave the concept of "fuzzy topology". He did the fuzzification of topology by replacing (subsets) in the definition of topology by (fuzzy sets) and introduced what we call Chang's fuzzy topological space. After that in 1976, Lowen [17] gave a modified definition of fuzzy topology by adding one simple condition, and made what we call Lowen's fuzzy topological space.

In this chapter we will introduce the basics of fuzzy topology,and then with some development starting with Chang's and Lowen's definitions and ending with another type of topological spaces called the intuitionistic fuzzy topological space [8]. Also in this chapter we will consider the openness and Closedness of fuzzy sets, besides, closure,interior, neighborhoods and those concepts over the product fuzzy topological spaces.

2.1 Definitions of Fuzzy Topological Spaces

Definition 2.1.1 (Chang):[6]:

Let I denote the unit interval [0, 1],and let X be a non-empty set,the set I^X of all fuzzy functions from X to I are the fuzzy subsets of X denoted by F(X). A fuzzy topology on a set X is a family $\tau \leq$ F(x) satisfying the following conditions:

(i) $\bar{1},\bar{0} \in \tau$

(ii) if A,B $\in \tau$ then A Λ B $\in \tau$

(iii) if { A_α: $\alpha \in$ index set Δ } is a family of fuzzy sets in τ then

$VA_\alpha \in \tau$, where $\alpha \in \Delta$.

The pair (X, τ) is called a C-fuzzy topological space and the members of τ are called the C-open fuzzy sets and their complements are called the C-closed fuzzy sets.

Later on, Lowen[17] defined the fuzzy topology on X as Chang did,but replaced the first condition (namely $\bar{\mathbf{1}},\bar{\mathbf{0}} \in \tau$) by all constant Fuzzy subsets $\bar{\boldsymbol{c}}$ where $\bar{\boldsymbol{c}}$(x) = c for all c∈[0,1]. Which is finer than Chang's topology.

Definition 2.1.2 (Lowen) [17]

Let *I* denote the unit interval [0,1],and let X be a non-empty set,the set I^X of all fuzzy functions from X to I are the fuzzy subsets of X denoted by F(X). A fuzzy topology on a set X is a family τ of fuzzy subsets of X satisfying the following conditions:

(i) all constant functions $\bar{c}$ from X to [0,1] ∈ τ.

(ii) if A,B $\in \tau$ then A Λ B $\in \tau$.

(iii) if { A_α: $\alpha \in$ index set Δ } is a family of fuzzy sets in τ then

$\vee A_\alpha \in \tau$; where $\alpha \in \Delta$.

Then the pair (X, τ) is called an L-fuzzy topological space and the members of τ are called the L-open fuzzy sets and their complements are called the L-closed fuzzy sets.

In the coming material we will use Chang's definition of fuzzy topology and call it the fuzzy topology, and if we use Lowen's definition we will use the notation L-fuzzy topology.

Examples of fuzzy topological spaces are parallel to those in the regular topological spaces:

for example,the indiscrete fuzzy topology $\{\bar{\mathbf{1}},\bar{\mathbf{0}}\}$ on X,

the discrete fuzzy topology on X, which consists of all fuzzy sets in X,

and the se of all crisp fuzzy sets in X is also a fuzzy topology.

Section (2):

2.2 Interior and Closure of Fuzzy Subsets

Definition:2.2.1:[6]

Let (X, τ)be a fuzzy topological space and let A be any fuzzy subset of X then:

(i) The closure of A denoted by $\bar{A}$ or Cl(A) is defined by:

$Cl(A) = \Lambda \{ F : F^c \in \tau : A \leq F \}$

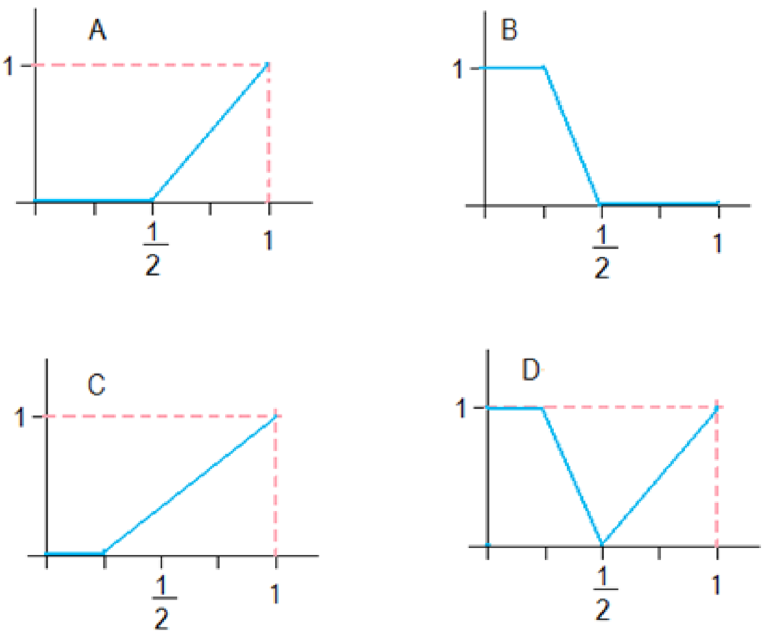
(ii) The interior of A denoted by A^o or int(A) is defined by:

$A^o = V \{ U : U \in \tau : U \leq A \}$

We will consider some examples to compute the closure and the interior of some fuzzy sets in a fuzzy topological space:

Example 2.2.2:

Given the following fuzzy sets A,B,C and D (fuzzy subsets of X=[0,1])

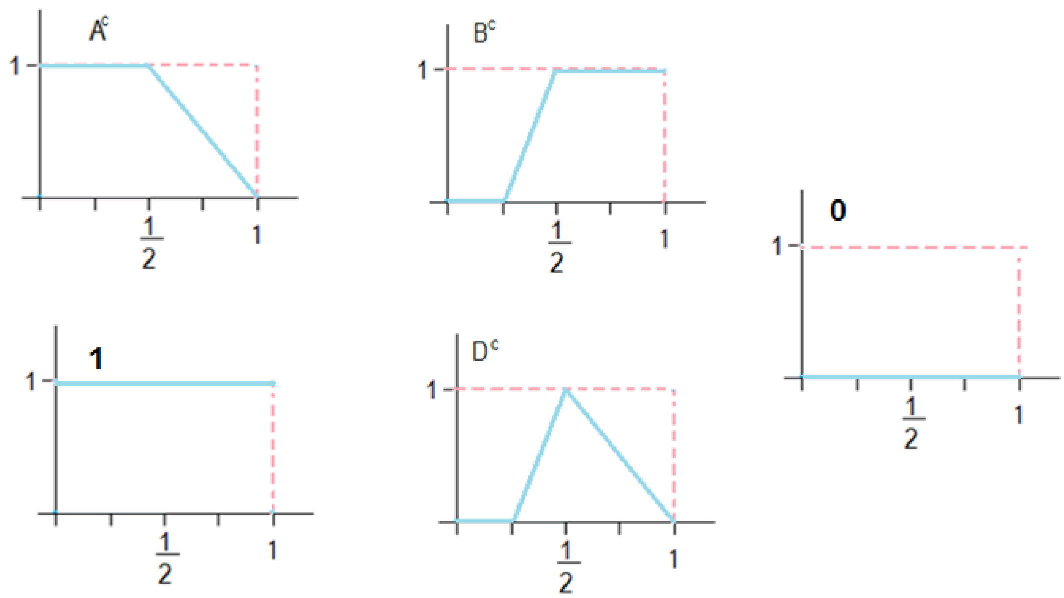


Where $\tau = \{\bar{\mathbf{0}}, \bar{\mathbf{1}}, A, B, D\}$

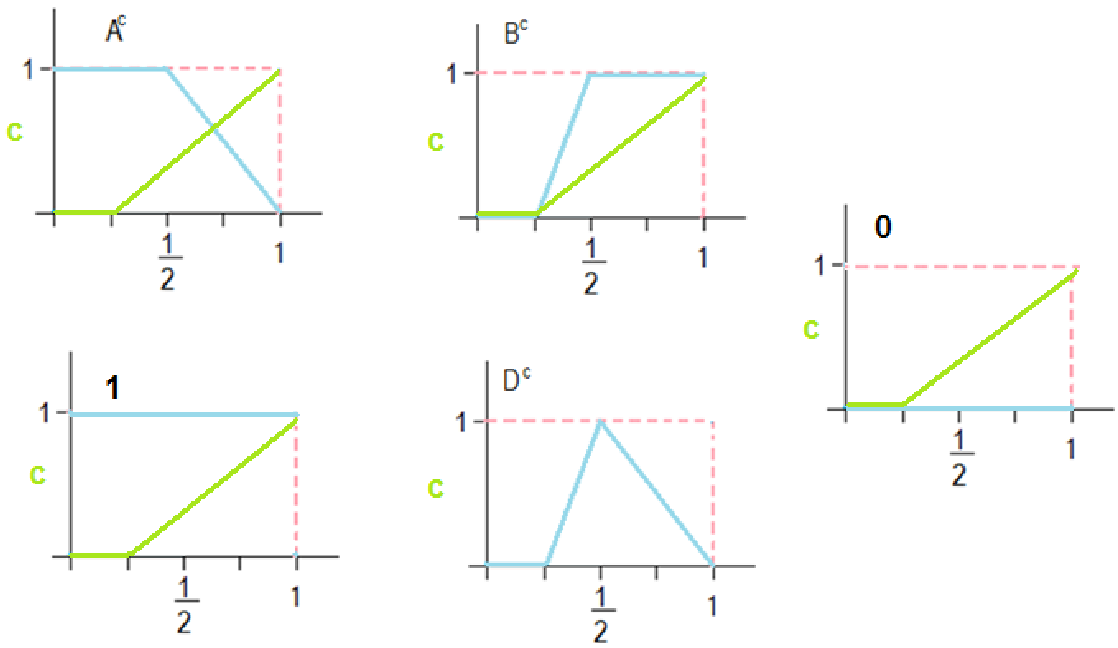
To find Cl(C) and C^o

First of all we find the fuzzy closed sets which are the complements of the members of τ.

The closed sets are:

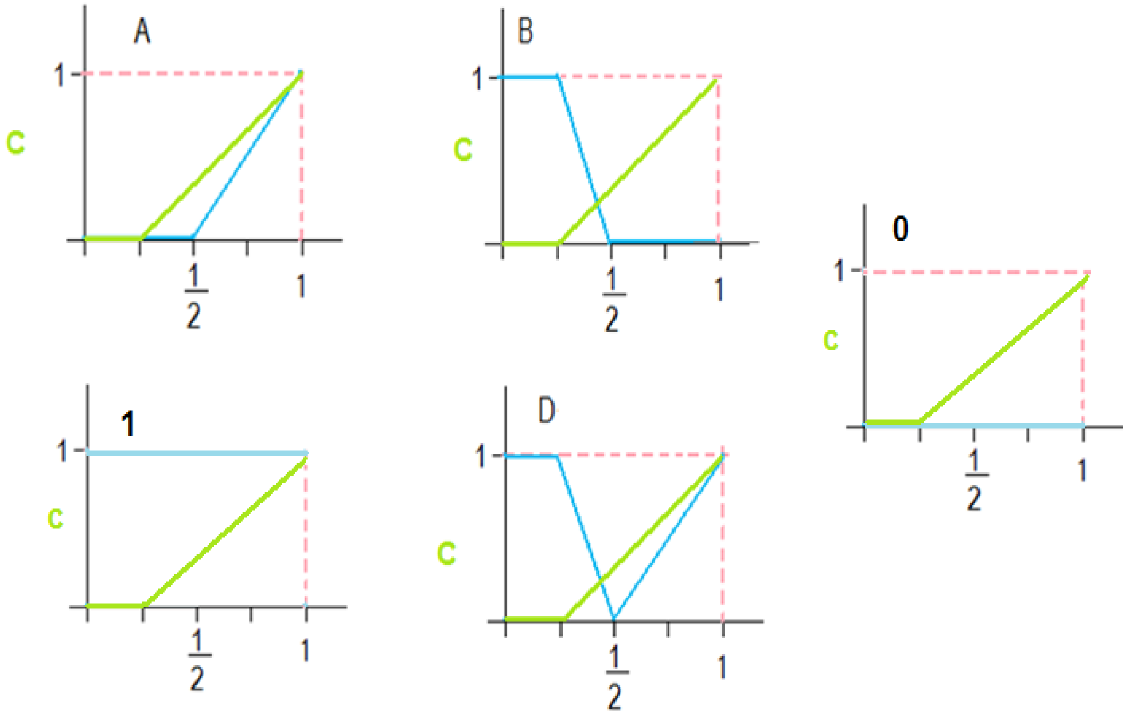


The fuzzy closed sets containing C are



And therefore $Cl(C)=\bar{1} \wedge B^c = B^c$.

For the interior consider the fuzzy open sets contained in C:



And therefore $C^o = A \vee \bar{\mathbf{0}} = A$

Example 2.2.3:

Let τ be the topology generated by A,B and C such that:

A= $\{ a_{0.7}, b_0, c_1 \}$,B= $\{ a_{0.7}, b_{0.5}, c_{0.3} \}$ and C= $\{ a_{0.5}, b_{0.5}, c_{0.5} \}$

To find Cl(A) and B^o,

Now, $\tau = \{ \bar{\mathbf{1}}, \bar{\mathbf{0}}, A, B, C, \{ a_{0.7}, b_{0.5}, c_1 \}, \{ a_{0.7}, b_{0.5}, c_{0.5} \},$

$\{ a_{0.7}, b_0, c_{0.3} \},\{ a_{0.5}, b_0, c_{0.5} \}, \{ a_{0.5}, b_{0.5}, c_{0.3} \}, \{ a_{0.5}, b_0, c_{0.3} \} \}$

The fuzzy closed sets are:

{ $a_{0.3}$, b_1, c_0 }, { $a_{0.3}$, $b_{0.5}$, $c_{0.7}$ }, { $a_{0.5}$, $b_{0.5}$, $c_{0.5}$ }, { $a_{0.3}$, $b_{0.5}$, c_0 }

{ $a_{0.3}$, $b_{0.5}$, $c_{0.5}$ }, { $a_{0.3}$, b_1, $c_{0.7}$ },{ $a_{0.5}$, b_1, $c_{0.5}$ }, { $a_{0.5}$, $b_{0.5}$, $c_{0.7}$ }

{ $a_{0.5}$, b_1, $c_{0.7}$ }, $\bar{\mathbf{1}}$, $\bar{\mathbf{0}}$.

Hence, Cl(A)= $\bar{\mathbf{1}}$,and $B^o = B$

Lemma 2.2.4: [37]

let τ be a fuzzy topology on X, then for any A,B fuzzy subsets of X the following are true:

1) $\overline{A \vee B} = \bar{A} \vee \bar{B}$

2) $(A \vee B)^o \geq A^o \vee B^o$

3) $(A^o)^c = \overline{A^c}$

4) $(\bar{A})^c = (A^c)^o$

Proof:

1) $\overline{A \vee B} = \bigwedge_{\substack{F\ closed \\ F\ \geq A \vee B}} F$

But $\bar{A} \vee \bar{B} = (\bigwedge_{\substack{F\ closed \\ F \geq A}} F) \vee (\bigwedge_{\substack{K\ closed \\ K \geq B}} K)$

$= \bigwedge_{\substack{F_i \geq A \\ K_j \geq B \\ F_i\ closed \\ K_j\ closed}} (F_i \vee K_j)$

$= \bigwedge_{\substack{L_t \geq A \vee B \\ L_t\ closed}} L_t = \overline{A \vee B}$

2) $A \leq A \vee B$ implies that $A^o \leq A \vee B$ And $B \leq A \vee B$ implies that $B^o \leq A \vee B$ So, $A^o \vee B^o \leq A \vee B$, but $A^o \vee B^o$ is open. Hence $A^o \vee B^o \leq (A \vee B)^o$.

3) $1-(A^o) = 1- \vee \{B: B \in \tau, B \leq A\}$

$= \wedge \{1-B: B \in \tau, B \leq A\}$

$= \wedge \{1-B: B \in \tau, 1-B \geq 1-A\}$

$= \wedge \{F: F^c \in \tau, F \geq 1-A\} = \overline{A^c}$.

4) $1-(\overline{A}) = 1- \wedge \{D: 1-D \in \tau, D \geq A\}$

$= \vee \{1-D: 1-D \in \tau, D \geq A\}$

$= \vee \{E: E \in T, E \leq 1-A\} = (1-A)^o = (A^c)^o$

Lemma 2.2.5 [37]

let $\{A_\alpha\}$ be the family of fuzzy subsets of a fuzzy space X then:

(i) $\vee \overline{A_\alpha} \leq \overline{\vee A_\alpha}$

(ii) $\vee \overline{A_\alpha} = \overline{\vee A_\alpha}$ where $\alpha \in$ finite indexing set

(iii) $\vee (A_\alpha)^o \leq (\vee A_\alpha)^o$

Proof:

(i) $\vee_\alpha \overline{A_\alpha} = \vee \{\wedge_\beta F_{\alpha_\beta}: F_{\alpha_\beta} \geq A_\alpha, F^c_{\alpha_\beta} \in \tau\}$

$= \wedge_\alpha \{\vee_\beta F_{\alpha_\beta}: \vee_\beta F_{\alpha_\beta} \geq \vee A_\alpha, F^c_{\alpha_\beta} \in \tau\}$

$\leq \wedge_\alpha \{ \vee_\beta F_{\alpha_\beta} : \vee K \geq \vee A_\alpha, (\vee K)^c \in \tau \}$

$= \overline{V_\alpha(A_\alpha)}$

(ii) In case of $\alpha \in \{1, 2, 3, \ldots, n\}$, then for (i) $(V\ F_{\alpha_\beta})^c \in \tau$ and equality holds.

(iii) To show $V\ (A_\alpha)^o \leq (V\ A_\alpha)^o$

$V\ (A_\alpha)^o = V_\alpha \{ \vee_\beta U_\beta : U_\beta \leq A_\alpha, U_\beta \in \tau \}$

$= V_\alpha \{ \vee_\beta U_\beta : V_\beta U_\beta \leq \vee_\alpha A_\alpha, V_\beta U_\beta \in \tau \}$

$\leq V_\alpha \{ \vee_\gamma : \vee_\gamma \leq \vee_\alpha A_\alpha, \vee_\gamma \in \tau \} = (V\ A_\alpha)^o$

Theorem 2.2.6:

Let (X, τ) be any fuzzy topological space, a fuzzy subset A of X is fuzzy closed if and only if $A = \bar{A}$

Proof:

Assume that $A = \bar{A}$. Since $\bar{A} = \wedge_{F\ closed} F \Rightarrow \bar{A}$ is fuzzy closed, and therefore A is fuzzy closed.

Conversely, Assume A is fuzzy closed.

$\bar{A}$ (x) = inf { F(x): F is fuzzy closed and A(x) ≤ F (x) },

that is $\bar{A}$ (x) ≤ F(x),consequently $\bar{A}$ (x) ≤ A(x) for every x in X.

Now, since $\bar{A} = \wedge$ { F: F is fuzzy closed and F ≥ A} then A(x) ≤ F(x) for every F ≥ A, therefore A(x) is a lower bound for the set { F(x): F ≥ A } and A(x) ≤ inf { F(x): F ≥ A }, therefore A(x) ≤ $\bar{A}$ (x) for every x in X.

Hence $\bar{A}$ (x) = A(x) for every x in X, which means $\bar{A} = A$.

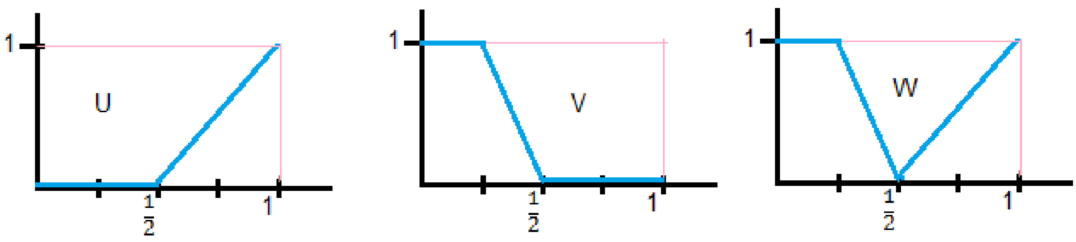
Definition 2.2.7:

Let (X,τ) be a fuzzy topological space, and let A be a fuzzy subset of X, we say a fuzzy point p is a fuzzy cluster point of A if for every nbd U of p, U ∧ A ≠ ϕ.

We show in the following example, the property that " if every neighborhood of a point intersects a set A implies that the point is in the closure of A" is not valid in fuzzy setting. The following example explains that:

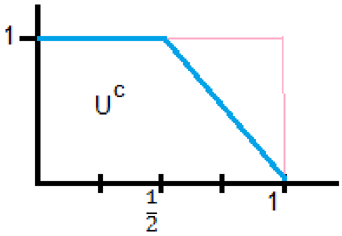
Example 2.2.8:

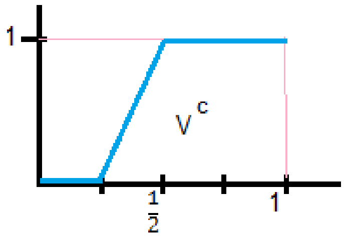
let X =[0,1] and let U,V and W be defined as follows:

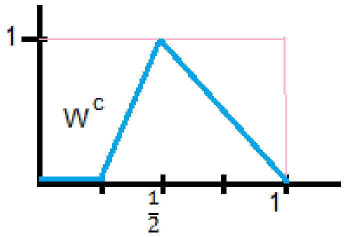


now, take $\tau = \{ \bar{0}, \bar{1}$,U, V, W }.

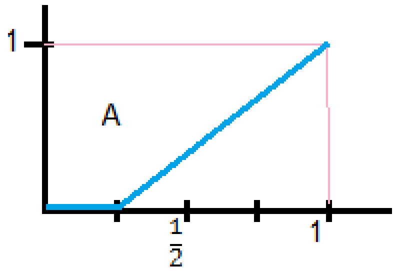
Therefore the complements of the fuzzy open sets are







Let p be the fuzzy point $\frac{1}{5}_{\frac{1}{4}}$ and let A be a fuzzy subset of X, as follows:



Then $\overline{A}$ = V^c, and the only neighborhoods of p are U, $\overline{1}_x$

It is clear that U $\wedge$ A $\neq \emptyset$ and $\overline{1}_x \wedge$ A $\neq \emptyset$ but still p $\notin \overline{A}$

Other types of fuzzy open and fuzzy closed sets were studied through research. Among them,

Definition 2.2.9 [37]:

a fuzzy set A in X is fuzzy regularly open if A= int (Cl(A))

a fuzzy set B in X is fuzzy regularly closed if B= Cl (int(B))

In the following we will define what is called a Q-neighborhood of a fuzzy point, which is used very often to deal with fuzzy topological concepts.

Definition 2.2.10 [37]:

A is a Q-neighborhood of a fuzzy point p if there exists B ∈ τ such that p Q B and B ≤ A.

We also define a fuzzy open (closed) mapping as an extension of non fuzzy setting as follows:

Definition: 2.2.11: [34]

Let $\bar{f}$:(X, τ)→ (Y,σ) be a mapping between fuzzy topologies. Then $\bar{f}$ is called:

(i) fuzzy open if and only if $\bar{f}$(u) ϵ σ for each u ϵ τ.

(ii) fuzzy closed if and only if $(\bar{f}(u))^c$ ϵ σ for each u^c ϵ τ

Finally, we define a semiopen (semiclosed) fuzzy sets parallel to non fuzzy setting as follows:

Definition 2.2.12:[37]

Let (X, τ) be a fuzzy topological space. For any fuzzy subset A of X, we say A is fuzzy semiopen if there exist U ϵ τ such that: $U \leq A \leq \overline{U}$

And for any fuzzy subset B of X we say B is fuzzy semiclosed if there exist F fuzzy closed such that: $F^o \leq B \leq F$

It is obvious that if A is fuzzy open then it is also a fuzzy semiopen set.

2.3 Fuzzy Membership and Neighborhood System

In studying the fuzzy open subsets of X we deal with the so called neighborhood system of fuzzy points. Also in studying separation axioms we deal with fuzzy points and fuzzy neighborhood systems; where ‘ the belonging ‘ between fuzzy points and fuzzy sets is greatly used. In 1974, C.K Wong [34] started the ‘’belonging of fuzzy point to a fuzzy set ‘’ concept. later on, different definitions of the same concept were added by Piu and Liu [22] M. Sarkar [27], Srivastava [32] and Wong [36]. These definitions were given independently. At the first look the definitions seem to be the same, but, after investigation they are found to be different in many aspects.

For the notation of fuzzy points we may use $p=x_\lambda$, or $p= (x,\lambda)$. Using the notation $p=a_t = \{ (x,t)\text{:where } t=0 \Leftrightarrow x \neq a\}$ it is clear that:

$A= V \{ (x, \lambda): 0 < \lambda \leq A(x): x\in \text{supp}(A) \}$

Also, we may write $A= V\ p: p \leq A$

That is A can be written as the union of its fuzzy points.

In the coming definition we classify the different definitions of the relation ‘ϵ’.

Definition 2.3.1:

Let A be a fuzzy subset of X, and let x_λ be a fuzzy point of X, we define the membership between x_λ and A as follows:

(i) $x_\lambda \in_1 A$ if and only if $\lambda < A(x)$

(ii) $x_\lambda \in_2 A$ if and only if $\lambda \leq A(x)$

(iii) $x_\lambda \in_3 A$ if and only if $\lambda = A(x)$

These definitions are essentially distinct.The following remarks show why:

Remark (1) 2.3.2:

$x_\lambda \in A V B$ if and only if $x_\lambda \in A$ or $x_\lambda \in B$,

which is true for all the definitions of "the belonging"

For $\in_1$: $x_\lambda \in_1 AVB$ means $\lambda < \max \{ A(x), B(x) \}$

So $\lambda < A(x)$ or $\lambda < B(x) \Leftrightarrow x_\lambda \in_1 A$ or $x_\lambda \in_1 B$

The same will be true if we replace $\in_1$ by $\in_2$ and $\in_3$.

Remark (2) 2.3.3:

$x_\lambda \in A \Lambda B$ if and only if $x_\lambda \in A$ and $x_\lambda \in B$.

This is true for all the definitions of the belonging ($\in_1$, $\in_2$ and $\in_3$)

The proof is similar to the that in remark (1).

Remark (1) and Remark (2) can be extended to any finite number of fuzzy sets $A_1, A_2, A_3, \ldots, A_n$.

In the case of arbitrary families of fuzzy sets { A_α, $\alpha\in\Delta$}, we have the following Lemma:

Lemma: 2.3.4:

Let { A_α: $\alpha\in\Delta$} be a family of fuzzy subsets then

(1) if $x_\lambda \in_1 \wedge A_{\alpha,}$ then $x_\lambda \in_1 A_\alpha$ for all $\alpha\in\Delta$

(2) $x_\lambda \in_1 \vee A_\alpha$ if and only if $x_\lambda \in_1 A_\alpha$ for some $\alpha\in\Delta$

proof:

(1) if $x_\lambda \in_1 \wedge A_\alpha$ then $\lambda < \inf \{ A_\alpha (x): \alpha\ \alpha\in\Delta$

so $\lambda < A_\alpha(x)$ for all $\alpha\in\Delta$

hence $x_\lambda \in_1 A_\alpha$ for all $\alpha\in\Delta$

(2)

Let $x_\lambda \in_1 A_\alpha$ for some $\alpha\in\Delta$ then $\lambda< A_\alpha (x)$ for some $\alpha\in\Delta$

so $\lambda < \sup \{ A_\alpha (x): \alpha\in\Delta \}$,then $\lambda <(\vee A_\alpha)(x)$ for $\alpha\in\Delta$

hence, $x_\lambda \in_1 \vee A_\alpha$

conversely; let $x_\lambda \in_1 \vee A_\alpha$

then $\lambda < \sup \{ A_\alpha (x): \alpha\in\Delta \}$,call this sup, *S*. so $\lambda<S$

take $\epsilon = \frac{S-\lambda}{2}$,as *S* is the sup, there exist one A_α (x) say $A_m(x)$

such that: $\lambda < A_m(x) < S$, hence $x_\lambda \in_1 A_m(x)$.

the converse of (1) in lemma 2.3.4 may not be true, as the following example shows:

Example 2.3.5:

Let X= {a, b, c}

$A_i = \{ a_{.2+10^{-(i+1)}} , b_{.3+10^{-(i+1)}}, c_{.2} \}$

i.e: $A_1 = \{ a_{.21} , b_{.31}, c_{.2}\}$

$A_2 = \{ a_{.201} , b_{.301}, c_{.2}\}$

$A_3 = \{ a_{.2001} , b_{.3001}, c_{.2}\}$

$\vdots \quad \vdots$

So $\wedge A_i$,over i, is equal to $\{ a_{.2} , b_{.3}, c_{.2}\}$

Hence, $b_{.3} \in_1 A_i$ for all i,but $b_{.3} \notin_1 \wedge A_i$

Replacing, $\in$ by $\in_2$, we have the following Lemma:

Lemma 2.3.6:

(1) $x_\lambda \in_2 \wedge A_\alpha \Leftrightarrow x_\lambda \in_2 A_\alpha$ for all α.

(2) if $x_\lambda \in_2 A_\alpha$,for some α then $x_\lambda \in_2 \vee A_\alpha$

proof: straight forward.

The converse of (2) may not be true,we may have $x_\lambda \in_2 \vee A_i$ but $x_\lambda \notin_2 A_i$, for all i, as the following two examples show:

Example (1):

Let X = {a}, $A_i = \{ a_{\frac{i}{i+1}} \}$, i = 1,2,3,… Then $\vee A_i = \{ a_1 \}$,We notice that $a_1 \in_2 \vee A_i$ but $a_1 \notin_2 A_i$, for all i.

Example(2):

Let X= {a, b}

$A_1 = \{ a_{0.49}, b_{0.3} \}$

$A_2 = \{ a_{0.499}, b_{0.3} \}$

$A_3 = \{ a_{0.4999}, b_{0.3} \}$

⋮

$\vee A_i = \{ a_{0.5}, b_{0.3} \}$

Now, $a_{0.5} \in_2 \vee A_i$ but $a_{0.5} \notin_2 A_i$ for every i.

Remark 2.3.7:

When replacing $\in$ by $\in_3$, then neither of these two statements are true:

(1) $x_\lambda \in_3 \wedge A_\alpha \Rightarrow x_\lambda \in_3 A_\alpha$ for all α.

(2) $x_\lambda \in_3 A_\alpha$, for some $\alpha \Leftrightarrow x_\lambda \in_3 \vee A_\alpha$

the following counter examples explain:

(i) Let X= {a}, take the fuzzy subsets

$A_1 = \{a_{0.51}\}$, $A_2 = \{a_{0.501}\}$, $A_3 = \{a_{0.5001}\}$, …

$\wedge_i A_i = \{a_{0.5}\}$

Clearly $a_{0.5} \in_3 \wedge_i A_i$ but $a_{0.5} \notin_3 A_i$ for all i.

(ii) Let X = {b}, take the fuzzy subsets

$B_1 = \{b_{0.39}\}$, $B_2 = \{b_{0.399}\}$, $B_3 = \{b_{0.3999}\}$, …

$\vee_i B_i = \{b_{0.4}\}$

Clearly, $b_{0.4} \in_3 \vee_i B_i$ but $b_{0.4} \notin_3 B_i$ for any i

Also, $b_{0.39} \in_3 B_1$ but $b_{0.39} \notin_3 \vee_i B_i$

In the case of crisp fuzzy subsets we could not use $\in_1$ because1 could not be smaller than A(x) for any $x \in X$.But we may use $\in_2$ and $\in_3$

We may look at a fuzzy point x_λ as a fuzzy subset B,therefore $x_\lambda \in A$ is equivalent to $B \leq A$, and $\in = \in_2$ satisfies this situation, so; it is appropriate to use $\in_2$ for $\in$.

Definition 2.3.8 [22]:

let (X,τ) be a fuzzy topological space, we say that a fuzzy set G is a neighborhood (nbd in short) of a fuzzy point $x_\lambda \Leftrightarrow$ there exist a fuzzy open set U such that $x_\lambda \leq U \leq G$.

The family of all neighborhoods of a fuzzy point x_λ is the neighborhood system of x_λ.

Theorem 2.3.9:[23]

let A be a fuzzy set in a fuzzy topological space (X,τ) then:

A is fuzzy open $\Leftrightarrow$ for each fuzzy point p= $x_\lambda \in A$, A is a nbd of p.

Proof:

$\Rightarrow$ Trivial

$\Leftarrow$ For each $x_\lambda \in A$, there exists U fuzzy open such that $x_\lambda \leq U \leq A$.

Therefore, $V\, x_\lambda \leq V\, U \leq A$. But, $A = V\, \{ x_\lambda : x_\lambda \in A \}$, therefore,

$A = V\{U: U$ is fuzzy open$\}$. Hence, A is fuzzy open.

The characterization of the fuzzy open sets using the neighborhoods of its fuzzy points, generates a topology.

Now, we consider a new neighborhood system, called the Q-neighborhood system.

Definition 2.3.10 [22]:

Let (X,τ) be a fuzzy topological space, let p= x_λ be a fuzzy point in X, We say that the fuzzy set A is a Q-neighborhood of p if there exists B $\epsilon\, \tau$,such that p Q B and $B \leq A$.

The family of all Q-neighborhoods of x_λ is called the system of Q-neighborhoods of x_λ.

Remark 2.3.11:

(1) The fuzzy set A is a Q-neighborhood of x_λ;doesn't mean that $x_\lambda \in A$ In 1916 Fréchet studied the neighborhood structure of neighborhoods (in non fuzzy settings) that doesn't contain the point itself, and it seems that the Q-neighborhood is an extension of that concept. In Fréchet work, dealing with regular sets A and A^c, we have $A \wedge A^c = \emptyset$ which is not the case in the fuzzy setting. However, in fuzzy settings A and A^c are not quasi coincident to each other.

The following theorem characterizes the properties of Q-neighborhood system:

Theorem 2.3.12: [22]

Let N(p) be the family of all Q-nbds of a fuzzy point $p = x_\lambda$

That is, N(p) = { U Q-nbd of p: p∈U }, we have the following:

i) if U ∈ N(p), then p Q U.

ii) If $U_1, U_2 \in N(p)$ then $U_1 \wedge U_2 \in N(p)$.

iii) U∈N(p),if $U \leq V$ then V∈N(p)

iv) If U ∈ N(p), then there exists V∈N(p) such that $V \leq U$ and for

every q Q V, V∈N(q).

Definition 2.3.13:

If $\{A_\alpha\}$ is a family of fuzzy sets in X, we say x_λ Q $\vee A_\alpha$ if and only if there exist A_{α_0} such that x_λ Q A_{α_0}

Fuzzy base and fuzzy subbase:

Definition 2.3.14:

A subfamily β of τ is called a fuzzy base or a fuzzy basis for (X, τ) if and only if each member of τ can be written as a union of members of β.

That is, for every A∈ τ, A= V b: for some b∈ β.

Definition 2.3.15:

a subfamily S of τ is called a subbase for τ if the collection of all finite intersections of members of S is a base for τ. That is, $\{ \wedge_{i=1}^{n} s$: s∈S } forms a fuzzy base for τ.

2.4 Fuzzy Product Topology

We define the fuzzy product topology on X xY using the fuzzy topologies on X and Y; as follows:

Definition 2.4.1: [cf 9, 1]

Let τ_X and τ_Y be two fuzzy topologies on X and Y respectively, the fuzzy product space is the cartesian product XxY with the fuzzy topology τ_{XxY} generated by the subbasis

$\{ p_1^{-1}(A_\alpha) \wedge p_2^{-1}(B_\beta): A_\alpha \in \tau_X, B_\beta \in \tau_y\}$ where:

p_1: is the projection function of XxY onto X, and

p_2: is the projection function of XxY onto Y.

Since $p_1^{-1}(A_\alpha) = A_\alpha \times \bar{1}_y$ and $p_2^{-1}(B_\beta) = \bar{1}_x \times B_\beta$, then the intersection

$p_1^{-1}(A_\alpha) \wedge p_2^{-1}(B_\beta) = (A_\alpha \times \bar{1}_y) \wedge (\bar{1}_x \times B_\beta) = A_\alpha \times B_\beta.$

Therefore, $B = \{ A_\alpha \times B_\beta : A_\alpha \in \tau_X, B_\beta \in \tau_Y \}$ forms a basis for τ_{XxY}.

The above definition of the fuzzy product topology on XxY can be extended to a finite family of fuzzy topological spaces $X_1, X_2, \ldots, X_n$

Let $X = \prod_{i=I}^{n} X_i$ be the fuzzy product space and P_i be the projection from X onto X_i, for each i=1,2,…,n.

If $B_i \in \tau_i$ then $p_i^{-1}(B_i)$ is a fuzzy set in X and $\{\wedge p_i^{-1}(B_i) : B_i \in \tau_i\}$ is

a subbasis that is used to generate a topology on X

This topology is called the fuzzy product topology for X

In the following, we will show the relationship between the product of the closure of fuzzy sets and the closure of the product.

First of all we will show that if A is a fuzzy closed set in X and B is a fuzzy closed in Y, then A x B is a fuzzy closed in X x Y

Theorem 2.4.2 [37]:

Let A and B be fuzzy closed subsets of X and Y respectively then A x B is a fuzzy closed subset of the fuzzy product space X xY

Proof:

If A is fuzzy closed in τ_x then A^c is fuzzy open in τ_x so(A^c x Y) is fuzzy open in XxY also since B is fuzzy closed in τ_y then B^c is fuzzy open in τ_y and (X x B^c) is fuzzy open in XxY.But, $(AxB)^c$ = 1- AxB

(1- AxB)(x,y) = 1 – min { A(x), B(y) }

= max { 1- A(x), 1- B(y) }

= max { $A^c(x)$, $B^c(y)$ }

= max { min { $A^c(x)$,1}, min {1, $B^c(y)$ } }

= max { (A^c x Y)(x,y), (X x B^c)(x,y)}

= (A^c x Y) V (X x B^c)

This is a union of two fuzzy open subsets in XxY so it is fuzzy open in XxY.

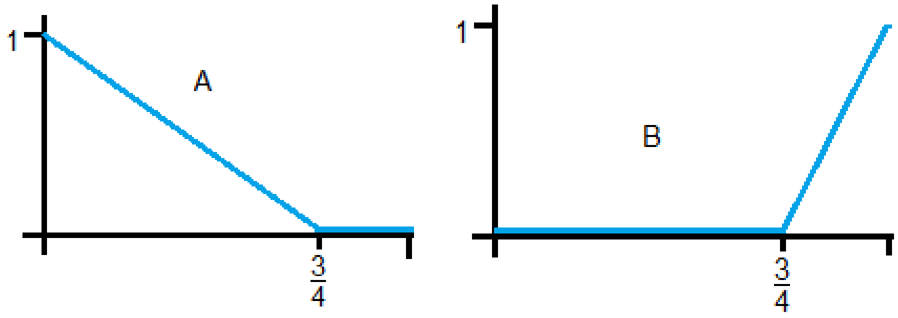
Since 1- AxB is fuzzy open in XxY, then AxB is fuzzy closed in XxY

Recall that in the usual set topology, it is true that the closure of the product is equal to the product of the closures, but,this is not the case in fuzzy setting.

The following example explains that: $\bar{A}$ x $\bar{B} \neq \overline{A \times B}$

Example: 2.4.3

Let A and B be as follows:



With

τ_x= {$\bar{0}$, $\bar{1}$, A^c } and τ_y= {$\bar{0}$, $\bar{1}$, B^c }

Now

$\bar{A}$=1 and $\bar{B}$=1 so $\bar{A}$ x $\bar{B}$ = 1

Now for $\overline{A \times B}$;

1 – A x B = (A^c x 1) V (1 x B^c)

Which is a union of two fuzzy open subsets of X x Y so it is fuzzy open in X x Y implying that Ax B is fuzzy closed in Xx Y, which then implies that $\overline{A \times B}$ = A x B ≠ $\bar{1}_{XxY}$

but $\bar{A}$ x $\bar{B}$ = 1,hence $\overline{A \times B} \neq \bar{A}$ x $\bar{B}$

In general the fuzzy closure of the products is a subset of the product of fuzzy closures, also, the product of the fuzzy interiors is a subset of the fuzzy interior of the products,the following theorem assures that:

Theorem 2.4.4 [37]:

let A be a fuzzy subset of X, and B be a fuzzy subset of Y then:

(i) $\bar{A}$ x $\bar{B} \geq \overline{A \times B}$

(ii) A^o x $B^o \leq (AxB)^o$

Proof:

(i) $\bar{A}$ is fuzzy closed and $\bar{B}$ is fuzzy closed,so $\bar{A}$ x $\bar{B}$ is fuzzy closed, also $\bar{A} \geq$ A and $\bar{B} \geq$ B,so $\bar{A}$ x $\bar{B} \geq$ A x B

But $\bar{A}$ x $\bar{B}$ is fuzzy closed,hence $\bar{A}$ x $\bar{B} \geq \overline{A \times B}$

Similarly

(ii) int(A) is fuzzy open and Int(B) is fuzzy open, so int(A) x int(B) is fuzzy open,also Int(A) ≤ A and Int(B) ≤ B,

so, int(A) x int(B) ≤ A x B

but, int(A) x int(B) is fuzzy open,

hence, int(A) x int(B) ≤ int(A x B)

S. Saha [26] modified the definition of the product to be "product related to" in such a way that makes: the product of the fuzzy closures equals the fuzzy closure of the products, and as well the product of the fuzzy interiors equals the fuzzy interior of the products.

Definition 2.4.5: [26]

let X and Y be two fuzzy spaces, X is said to be "product related" to Y if for any C fuzzy subset of X, D fuzzy subset of Ysuch that:

if $U^c < C$ and $V^c < D \Rightarrow (U^c \, x1) \vee (1x \, V^c) \geq C x \, D$: $U \in \tau_x, V \in \tau_y$

Then there exist $U_1 \in \tau_x$ and $V_1 \in \tau_y$ such that:

$U_1^c \geq C$ or $V_1^c \geq D$ and $(U_1^c \, x1) \vee (1x \, V_1^c) = (U^c \, x1) \vee (1x \, V^c)$

Finally, Saha proved the following theorem

Theorem 2.4.6:[26]

Let X and Y be two fuzzy spaces, A be a fuzzy subset of X and B be a

fuzzy subset of Y, if X is product related to Y,then:

(i) $\bar{A}$ x $\bar{B}$ = $\overline{A \times B}$ and

(ii) A^o x B^o = $(AxB)^o$

2.5 The Intuitionistic Fuzzy Topological Space

The definition of the intuitionistic fuzzy subset was given for the first time by K. T. Atanassov [4], it generalizes Zadeh's concept of fuzzy subsets, then as an extension of the chang's definition of the fuzzy topological spaces,D. Çoker in [8] gave the definition of intuitionistic fuzzy topological spaces using the intuitionistic fuzzy sets.

It was followed by Mondal and Samanta [24] who introduced in 2002 the concept of intuitionistic gradation of openness also as an extension of gradation of openness given by Chattopadyay [7]. Min and Park in [20, 21] studied an equivalent form of the intuitionistic fuzzy topological space, where they defined the value of the components of the intuitionistic fuzzy sets by defining two functions from the fuzzy subsets on X to the unit interval [0,1].

Definition 2.5.1:[4]

Let X be a non empty set, an intuitionistic fuzzy set A (IFS) is defined to be the ordered pair $A = < A_1, A_2>$ where $A_1:X\rightarrow[0,1]$ and

$A_2:X\rightarrow[0,1]$ such that: $0 \leq A_1(x) + A_2(x) \leq 1$ for every x in X.

$A_1(x)$ denotes the degree of Membership of each element $x \in X$,

and $A_2(x)$ denotes the degree of nonmembership for each $x \in X$.

The intuitionistic fuzzy set $\tilde{0}=<\bar{0}, \bar{1},>$ is the empty intuitionistic fuzzy set, and $\tilde{1}=<\bar{1}, \bar{0},>$ is the whole intuitionistic fuzzy set.

The ordinary fuzzy set A can be written as $<A,A^c>$ as an IFS.

Let $A = < A_1,A_2 >$, $B = < B_1,B_2 >$ be two intuitionistic fuzzy sets

we say $A\subseteq B$ to mean $A_1 \leq B_1$ and $A_2 \geq B_2$ for each $x \in X$.

We define the complement of A (i.e A^c) to be $A^c = < A_2,A_1 >$.

The intersection and the union of A and B is defined by:

$A \cap B = < A_1 \wedge B_1, A_2 \vee B_2 >$, and $A \cup B = < A_1 \vee B_1, A_2 \wedge B_2 >$.

Of course; the intersection and the union could be extended

to any family of intuitionistic fuzzy sets

i.e. if $A_i = < A_{i_1}, A_{i_2} >$ then $\cap_i A_i = < \wedge A_{i_1}, \vee A_{i_2} >$ and

$\cup_i A_i = < \vee A_{i_1}, \wedge A_{i_2} >$.

We say $A = <A_1,A_2>$ and $B = <B_1,B_2>$ are intuitionistic Q-coincident (IQ-coincident) if and only if there exists $x \in X$ such that: $A_1(x) > B_2(x)$ or $A_2(x) < B_1(x)$.

Definition 2.5.2:

Let F(X) be the family of fuzzy subset of a non empty set X then IF(X) is the family of intuitionistic fuzzy subset of X.

Consider f: X→Y and $\bar{f}$: F(X) → F(Y) then:

f^*: IF(X) → IF(Y) is the intuitionistic fuzzy function

defined by $f^*(<A_1,A_2>) = < \bar{f}(A_1), (\bar{f}\,(A_2^c))^c >$

and $(f^*)^{-1}(<B_1,B_2>) = < \overline{(f)}^{-1}(B_1), \overline{(f)}^{-1}(B_2) >$

Coker in 1997 in [8] proved the following properties of images and preimages between intuitionistic fuzzy functions and intuitionistic fuzzy sets.

Theorem 2.5.3 [8]:

Let f: X → Y be a function, and $\bar{f}$: F(X) → F(Y) be a fuzzy function, and f^*: IF(X) → IF(Y) is the intuitionistic fuzzy function

Then for any A,B ∊ IF(X), C,D ∊ IF(Y), we have:

1) A ≤ B ⇒ f^*(A) ⊆ f^*(B), and C ≤ D ⇒ $(f^*)^{-1}$(C) ⊆ $(f^*)^{-1}$(D)

2) $(f^*)^{-1}(\widetilde{1_y}) = \widetilde{1_x}$, $(f^*)^{-1}(\widetilde{0_y}) = \widetilde{0_x}$

3) f^*(AVB) = f^*(A) ∪ f^*(B),and f^*(A ∧ B) = f^*(A) ∩ f^*(B)

4) $(f^*)^{-1}$(CVD) = $(f^*)^{-1}$(C) ∪ $(f^*)^{-1}$(D),

and, $(f^*)^{-1}$(C ∧ D) = $(f^*)^{-1}$ (C) ∩ $(f^*)^{-1}$ (D).

Now, we come to the definition of the intuitionistic fuzzy topological spaces, as an extension of Chang's fuzzy topological spaces.

Definition 2.5.4 [8]:

Let τ be a family of intuitionistic fuzzy sets on X satisfying the following properties:

1) $\tilde{0}_x \in \tau$, $\tilde{1}_x \in \tau$

2) If A,B ∊ τ then A ∩ B ∊ τ

3) Let { A_α: α∊Δ } be a family of intuitionistic fuzzy sets then $\cup_\alpha (A_\alpha) \in \tau$.

Then (X, τ) is called an intuitionistic fuzzy topological space denoted by IFTS.

Any member of τ is called an intuitionistic fuzzy open set, and its complement is called an intuitionistic fuzzy closed set.

Also for any intuitionistic fuzzy set A we define:

Iint(A) = $A^o = \cup$ { U: U is intuitionistic fuzzy open set: U⊆A }

and Icl(A) = $\overline{A}$ = ∩ { F: F is intuitionistic fuzzy closed set: F⊇A }

Coker also defined the fuzzy intuitionistic point as follows

Definition 2.5.5 [8]:

Let $a \in X$ be a fixed element and α∈(0,1] and β∈[0,1), where α+β ≤ 1,then $a_{(\alpha,\beta)} = < a_\alpha, a^c_{1-\beta} >$ is called an intuitionistic fuzzy point.

For any intuitionistic fuzzy set A = < A_1, A_2 > define

$a_{(\alpha,\beta)} \in A$ iff $\alpha \le A_1(a)$ and $\beta \ge A_2(a)$.

Also we define a vanishing intuitionistic fuzzy point $a_{(\beta)} = <\overline{0}, a^c_{1-\beta}>$ and $a_{(\beta)} \in A$ iff $0 \le A_1(a)$ and $\beta \ge A_2(a)$.

Any topological space can be considered as an intuitionistic fuzzy topological space. Because if (X,$\mathcal{F}$) is a topological space,

let τ = { < A, A^c >: A∈ $\mathcal{F}$ },then τ is an IFTS.

To show that τ is a topology:

- $\emptyset \in \mathcal{F} \Rightarrow <\emptyset, \emptyset^c> = <\emptyset, X> = <\bar{0}_{x}, \bar{1}_{x}> = \tilde{0}_{x}$, and $X \in \mathcal{F}$ implies that

  $<X, X^c> = <X, \emptyset> = <\bar{1}_{x}, \bar{0}_{x}> = \tilde{1}_{x}$,

  so $\tilde{0}_x$ and $\tilde{1}_x \in \tau$.

- If $A,B \in \mathcal{F}$ then $<A, A^c> \cap <B, B^c> = <A \wedge B, A^c \vee B^c>$

So $<A, A^c> \cap <B, B^c> = <A \wedge B, (A \wedge B)^c> \in \tau$, since $A \wedge B \in \mathcal{F}$.

- $\{ A_\alpha : \alpha \in \Delta \}$ a family of intuitionistic fuzzy sets $\in \mathcal{F}$

$\cup_\alpha < A_\alpha, A_\alpha^c> = < \cup_\alpha A_\alpha, \cap_\alpha A_\alpha^c> = < \cup_\alpha A_\alpha, (\cup_\alpha A_\alpha)^c > \in \tau$,

Since $\cup_\alpha A_\alpha \in \mathcal{F}$.

Chapter Three

Extending Separation Axioms for The Fuzzy Topology

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3.1 Fuzzy Hausdorff Spaces

Introduction

Fuzzy separation axioms including fuzzy Hausdorffness are basic concepts that have been studied by several authors and in different approaches: cf [21, 28, 30, 2, 31, 5, 3]Lowen [17] approach was by using his fuzzy convergence theory, while Pu and Liu [22] approach used Q-relation concept, and Srivastava [32] approach used fuzzy points. Those different approaches of Hausdorffness mainly look completely different but they turn out to be equivalent.

The following definition of fuzzy Hausdorff is the one used fuzzy points, which is parallel to the definition of Hausdorffness in regular setting of topological spaces.

Definition 3.1.1 [32]:

A fuzzy topological space (X,τ) is said to be Hausdorff if and only if for every distinct fuzzy points x_λ and y_r,there exist $U,V \in \tau$such that $x_\lambda \in U$, $y_r \in V$ and $U \wedge V = \Phi$.

The following two lemmas will be used to present the relationship between Quasi-coincident and membership for a fuzzy point and a fuzzy set

Lemma 3.1.2:

$x_{1-r} \in A \Leftrightarrow x_r \, Q \, A$

Proof:

$x_r \, Q \, A \Leftrightarrow r + A(x) > 1 \Leftrightarrow A(x) > 1-r$

$\Leftrightarrow 1-r < A(x) \Leftrightarrow x_{1-r} \in A$.

Lemma 3.1.3:[31]

If for any two distinct fuzzy points $x_{r,}$ y_s in X, there exists U,V $\in \tau$ such that $x_r \in U$, $y_s \in V$ and $U \wedge V = \Phi$,then for every x,y $\in$ X with $x \neq y$, there exists U,V $\in \tau$ such that U(x)>0,V(y)>0 and $U \wedge V = \Phi$.

Proof:

For any x, y $\in X$ with $x \neq y$, take any r $\in$ (0,1), s= 1- r then by assumption there exist U,V $\in \tau$ such that: $x_r \in U$, $y_{1-r} \in V$,$U \wedge V = \Phi$. Now, $x_r \in U$ implies that U(x) > r > 0 and $y_{1-r} \in V$ implies that V(y)> 1-r >0.

There is an equivalent definition of Hausdorff fuzzy topological spaces using the concept of Q-neighborhoods.

Theorem 3.1.4 [31]:

Let (X,τ) be a fuzzy topological space, then the following are equivalent:

(i) (X, τ) is hausdorff

(ii) for any distinct fuzzy singletons $x_{r,}$ y_s in X, there exist U Q-nbd of $x_{r,}$ V Q-nbd of $y_{s,}$such that $U \wedge V = \Phi$

Proof:

(i) $\Rightarrow$ (ii) Let x_r, y_s be fuzzy singletons in X, with $x \neq y$. So, there exist U,V $\in \tau$ such that $x_r \in U$, $y_s \in V$ and $U \wedge V = \Phi$

Then we have one of the following four cases

Case (1): $r<1$ and $s<1$

This implies that x_{1-r}, y_{1-s} are fuzzy points since $1-r \neq 0$, $1-s \neq 0$.

Then by the assumption, for x_{1-r} and y_{1-s},there exists U,V$\in \tau$ such that:$x_{1-r} \in U$, $y_{1-s} \in V$ and $U \wedge V = \Phi$.

Therefore, $1-r < U(x)$ and $1-s < V(y)$ which implies that $U(x) + r > 1$ and

$V(y) + s > 1$. Therefore by lemma(3.1.2), we get that x_r Q U and y_s Q V

and $U \wedge V = \Phi$.

Case (2): $r=1$, $s<1$

For any fixed $k \in (0,1)$, consider the fuzzy points x_k, y_{1-s},

By assumption there exist U,V $\in$T such that $x_k \in U$, $y_{1-s} \in V$, and $U \wedge V = \Phi$,but then $0<k<U(x)$ and $1-s<V(y)$, this implies $U(x)>1-r$ where $r=1$ and $V(y)+s>1$ Which means, x_r Q U and, y_s Q V

case (3): $r<1$, $s=1$. The prove is similar to that of case 2.

case (4): $r = s = 1$

using lemma(3.1.3), there exists U,V ∈T with U(x)>0 and V(y)>0 that is U(x) > 1-r,where r=1 and V(y) > 1-s,where s =1, or U(x) + r > 1 and V(y) + s > 1 which means x_r Q U and y_s Q V.

(ii)⇒ (i) given any pair of distinct fuzzy points x_r and y_s in X, take the fuzzy singletons x_{1-r}, y_{1-s} in X then by the condition, there exist U,V ∈ τ, U∧V =Φ, such that x_{1-r} Q U and y_{1-s} Q V which implies x_r ∈U,y_s ∈V and U∧V =Φ.

As long as we deal with fuzzy topological spaces on a fuzzy set X, which has α-levels, it is natural to define Hausdorffness using the α-levels, and it is called α-Hausdorff.

Definition 3.1.5 [31]:

A fuzzy topological space (X, τ) is said to be α-Hausdorff for α∈[0,1) if and only if for each x,y ∈ X with x ≠y, there are U,V ∈ τ such that: U(x)>α, V(y)>α and U∧V =Φ.

In the following theorem we introduce the relationship between Hausdorffness and α- Hausdorffness.

Theorem 3.1.6:[31]

A fuzzy topological space (X,τ) is Hausdorff if and only if it is α-hausdorff for every α∈[0,1)

proof:

case(1): $\alpha \in (0,1)$

take $x \neq y$, so x_α and y_α are distinct fuzzy points then there exist U,V $\in \tau$,such that $x_\alpha \in U$ and $y_\alpha \in V$,with $U \wedge V = \Phi$, which means there exist U,V $\in \tau$, such that $U(x)>\alpha$, $V(y)>\alpha$. Hence, X is α-Hausdorff.

case(2): $\alpha=0$

Take $x \neq y$.

Fix $r,s \in (0,1)$ and consider the fuzzy points x_r and $y_{s.,}$ there exist U,V $\in \tau$ such that $x_r \in U$ and $y_s \in V$,with $U \wedge V = \Phi$. That is, $0<r<U(x)$, $0<s<V(y)$, and so $U(x)>\alpha$, $V(y)>\alpha$ and $U \wedge V = \Phi$.

conversely,

Assume that (X,τ) is α-Hausdorff for every $\alpha \in [0,1)$. Let x_r and y_s be fuzzy points with $x \neq y$. If $r \leq s$ then since X is s-Hausdorff,

there exist U,V $\in \tau$ such that $U(x)>s$, $V(y)>s$ and $U \wedge V = \Phi$.

So $U(x)> s \geq r$, $V(y) > s$, and $U \wedge V = \Phi$. Hence, $x_r \in U$ and $y_s \in V$,and $U \wedge V = \Phi$.

A similar argument can be used if $r>s$.

Thus$(X, \tau)$is Hausdorff.

Now,we summarize the different definitions involving Hausdorffness in the following main theorem:

Theorem 3.1.7:

Let (X,τ) be a fuzzy topological space then the following definitions of Hausdorffness are equivalent:

(i) For every distinct fuzzy points x_λ and y_r,there exist $U,V \in \tau$ such that $x_\lambda \in U, y_r \in V$ and $U \wedge V = \Phi$.

(ii) For every distinct fuzzy singletons x_λ and y_r,there exist $U,V \in \tau$ such that x_λ Q U and y_r Q V and $U \wedge V = \Phi$.

(iii) For every $x,y \in X$, with $x \neq y$ and for every $\alpha \in [0,1)$ there exist $U,V \in \tau$ such that $U(x) > \alpha$ and $V(y) > \alpha$ and $U \wedge V = \Phi$.

3.2 Other Fuzzy Separation Axioms

Definition 3.2.1[13]:

A fuzzy topological space is said to be fuzzy-T_0 if and only if for any x_λ, y_s, two fuzzy singletons with $x \neq y$, there exists a fuzzy open set U, such that $x_\lambda \leq U \leq y_s^c$ or $y_s \leq U \leq x_\lambda^c$.

Definition 3.2.2[13]:

A fuzzy topological space is said to be fuzzy-T_1 if and only if for x_λ, y_s, two fuzzy singletons with $x \neq y$, there exist two fuzzy open sets U, V such that $x_\lambda \leq U \leq y_s^c$ and $y_s \leq V \leq x_\lambda^c$.

It is obvious that (X,τ) is fuzzy $T_1 \Rightarrow (X, \tau)$ is fuzzy T_0.

The following example shows a T_0 space may not be T_1.

Let X ={a,b }, τ = { $\bar{0}$, $\bar{1}$, $\{a_{0.9},b_{0.2}\}$, $\{a_{0.99},b_{0.2}\}$,$\{a_{0.999},b_{0.2}\}$,… }.

For any a_λ, b_r, there exist U: neighborhood of a_λ such that

$a_\lambda \in U \leq b_r^c = \{ a_1, b_{0.8} \}$. Therefore, τ is T_0.

But it is not T_1 by taking a_λ, $b_{0.3}$. There is no V∈τ such that

$b_{0.3} \in V \leq a_\lambda^c = \{ a_{1-\lambda}, b_1\}$.

In standard topological spaces the T_1 space was identified by the property that every singleton is closed. In fuzzy setting this property is not extended but a weaker condition could be used as the following theorem states.

Theorem 3.2.3 [13]:

A fuzzy topological space (X, τ) is T_1 if and only if every crisp singleton is closed (i.e. x_1 is closed for every x ∈X)

proof:

For the first direction let X be a non empty set and a ∈X,take a_1 to be any crisp singleton, now for any y_r:where y ∈X,y ≠a, r∈(0,1],consider the fuzzy singletons a_1 and y_r, since X is T_1

There exist two fuzzy open sets U,V in τ such that:$a_1 \in U \leq y_r^c$ and $y_r \in V \leq a_1^c$.

And since it is true for any r ∊ (0,1],then V(y) = sup { r: r∊(0,1] } =1

for y $\neq$a.

Also since V ≤ a_1^c then V(a) =a_1^c(a) = 1-1 = 0

that is V(x) = $\begin{cases} 0 & if\ x = a \\ 1 & if\ x \neq a \end{cases}$ which means V=a_1^c

but V is open, hence a_1 is closed.

For the other direction let x_λ, y_r with x $\neq$y be two distinct singletons, since every crisp singleton is closed then x_1, y_1 are closed sets.

Let U= y_1^c and V=x_1^c, then U and V are open, x_λ ∊ U ≤ y_1^c and y_r ∊ V ≤ x_1^c.Hence X is T_1.

We will modify the definition of T_1 space (namely strong T_1 space) to insure the validity of the property that every fuzzy singleton is closed as an extension of standard topological spaces.

Definition 3.2.4 [13]:

A fuzzy topological space is said to be fuzzy strong-T_1 (in short T_s) if and only if every fuzzy singleton is a closed fuzzy set.

An example of a T_s space:

Let X ={ a, b }

and τ = { $\overline{0}$, $\overline{1}$, { a_λ, b_1 }, { a_1, b_r }, { a_λ, b_r}: for every λ,r ∊ (0,1) }

then every fuzzy singleton is closed.

It is clear that if (X, τ) is fuzzy T_s then (X, τ) is fuzzy T_1.

Following the previous definitions of T_0 and T_1 spaces, a fuzzy T_2 space is defined as follows:

Definition 3.2.5 [13]:

A fuzzy topological space is said to be fuzzy-T_2 if and only if for any two fuzzy singletons x_λ, y_s with $x\neq y$, there exist two fuzzy open sets U, V such that $x_\lambda \leq U \leq y_s^c$ and $y_s \leq V \leq x_\lambda^c$ and $U \leq V^c$.

An example of a T_2 space is the following:

Example 3.2.6:

let X= {x,y }

and $\tau = \{\bar{0}, \bar{1}, \{ x_\lambda, y_0\}, \{ x_0, y_s\}, \{ x_\lambda, y_s\}: \lambda \geq \frac{1}{2}, s \geq \frac{1}{2} \}$

for any two distinct singletons x_t and y_r

case (1): $t < \frac{1}{2}$, $r < \frac{1}{2}$

Take $U = \{ x_{1-t}, y_0\}$ and $V= \{ x_0, y_{1-r}\}$

Now, $t < \frac{1}{2} \rightarrow 1-t > \frac{1}{2}$ so $t < 1-t$ which implies that $t < U(x)$

hence $x_t \in U$,and, $y_r^c = \{x_1, y_{1-r}\}$,so $U \leq y_r^c$.

Similarly, $r < \frac{1}{2} \rightarrow 1-r > \frac{1}{2}$, which means $r<1-r$ that is $y_r \in V$,

since $x_t^c = \{x_{1-t}, y_1\}$ so $V \leq x_t^c$ and $U= \{ x_{1-t}, y_0 \} \leq \{ x_1, y_{1-r}\} = V^c$.

case(2): x_t and y_r where $t \geq \frac{1}{2}$, $r \geq \frac{1}{2}$

Take U={ x_t, y_0} and V={ x_0, y_r}, then $x_t \in$ U and U≤{ x_1, y_{1-r} } = y_r^c

also, $y_r \in$V and V≤ { x_{1-t}, y_1 } =x_t^c and U={ x_t, y_0} ≤ { x_1, y_{1-r} } = V^c

case(3): x_t and y_r where $t < \frac{1}{2}$, $r \geq \frac{1}{2}$

Take U = { x_{1-t}, y_0} and V= { x_0, y_r}.

Since t<1-t, then $x_t \in$U and U ≤ y_r^c = { x_1, y_{1-r}}

also $y_r \in$ V and V ≤{ x_{1-t}, y_1} = x_t^c and U = { x_{1-t}, y_0} ≤ { x_1, y_r } = V^c

and therefore, this topological space is fuzzy T_2

Definition 3.2.7 [13]:

A fuzzy topological space (X, τ) is said to be fuzzy Urysohn (fuzzy -$T_{2\frac{1}{2}}$) if and only if for every, two fuzzy singletons x_λ, y_s with x≠ y,there exist two fuzzy open sets U,V such that: $x_\lambda \leq U \leq y_s^c$, $y_s \leq V \leq x_\lambda^c$ and cl(U)≤ $(cl(V))^c$.

It is easy to show that if (X, τ) is fuzzy $T_{2\frac{1}{2}}$ topological space then (X, τ) is fuzzy T_2 [13].

Definition 3.2.8 [13]:

A fuzzy topological space (X, τ) is called fuzzy regular space if and only if for every fuzzy singleton p=x_λ and fuzzy closed subset F of X such that p $\in F^c$, there exist U,V $\in \tau$ such that p $\in$U, F ≤V and U ≤ V^c

There is also an other equivalent definition for (X, τ) to be a regular space if and only if for every fuzzy singleton p=x_λ and fuzzy open subset U of X with x_λ ∈U, there exist V ∈ τ such that:p ∈ V ≤ $\bar{V}$ ≤ U.

Theorem 3.2.9 [13]:

Let (X, τ) be a fuzzy regular topological space, then for a fuzzy closed subset F of X and a fuzzy singleton p= x_λ where x_λ ∈ F^c there exist U, V ∈ τ such that x_λ ∈U, F≤ V and $\bar{U}$≤ $(\overline{V})^c$.

proof:

F is a fuzzy closed subset of X so F^c is open where p= x_λ∈ F^c. Then by the definition of the fuzzy regular space, there exist V∈ τ such that p ∈ V ≤ $\bar{V}$ ≤ U = F^c.

Take V = $(\overline{F^c})^c$ then $\bar{V}$ ≤ $(\overline{U})^c$ where U= F^c.

Now, we define a T_3 space.

Definition 3.2.10:

A fuzzy regular T_s topological space is called T_3 space.

Back to the classical topological spaces, if we have a T_0 space (X, τ) which is also regular, then (X, τ) is a T_3 space, but it is not the case of the fuzzy topological spaces, as shown in the next theorem:

Theorem 3.2.11[13]:

If (X, τ) is a T_0 and a regular space then it is a fuzzy Urysohn's space

Proof:

Let (X,τ) be a regular T_0 space and let $p=x_\lambda$ and $q=y_r$ be two fuzzy singletons with $x \neq y$, since (X, τ) is T_0 then there exist $U,V \in \tau$ s.t $x_\lambda \in U \leq y_r^c$. Take $F = U^c$, that is $U = F^c$ is open and $x_\lambda \in F^c$.

But by theorem [3.2.9] since F is closed subset of a regular space, there exist $V,W \in \tau$ such that $x_\lambda \in V$, $F \leq W$ and $\bar{V} \leq (\bar{W})^c$ but $y_r \in U^c = F \leq W$. Hence $x_\lambda \in V$, $y_r \in W$ and $\bar{V} \leq (\bar{W})^c$,so (X, τ) is $T_{2\frac{1}{2}}$ space (Urysohn).

Definition 3.2.12:

A fuzzy topological space (X, τ) is called normal space, if and only if for every fuzzy closed subsets F_1,F_2 of X such that $F_1 \leq (F_2)^c$ there exist $U,V \in \tau$ such that $F_1 \leq U$, $F_2 \leq V$ and $U \leq V^c$.

Definition 3.2.13:

A fuzzy normal T_s space (X, τ) is called T_4 space.

Theorem 3.2.14 [13]:

A closed subset of a normal space is normal.

Proof:

Let (X, τ_x) be a fuzzy normal topological space and let A be a closed subset of X,then (A, τ_A) is a subspace.

Take F_1, F_2 any two fuzzy closed subsets of A with $F_1 \leq A- F_2$, since A is fuzzy closed subset of X $\Rightarrow F_1 \leq X - F_2$, and since (X, τ_x) is normal then there exist U, V $\in \tau_x$ such that $F_1 \leq U$, $F_2 \leq V$ and $U \leq V^c$.

Now, $A \wedge U$ and $A \wedge V$ are two fuzzy open subsets of τ_A such that $F_1 \leq A \wedge U$, $F_2 \leq A \wedge V$ and $A \wedge U \leq A \wedge V^c = (A \wedge V)^c$.

3.3 Intuitionistic Fuzzy Separation Axioms

Recall that, if $\alpha \in (0,1]$ and $\beta \in [0,1)$ such that $\alpha + \beta \leq 1$ then for any a in X, $a_{(\alpha,\beta)}$ is an intuitionistic fuzzy point defined by:

$a_{(\alpha,\beta)} = < a_\alpha, a^c_{1-\beta} >$.

This means that a_α takes a to α and all other elements of X to 0 and

$a^c_{1-\beta}$ takes a to β and all other elements of X to 1.

Also, recall the intuitionistic vanishing fuzzy point $a_{(\beta)}$ where

$a_{(\beta)} =< \bar{0}, a^c_{1-\beta} >$.

we will now define the intuitionistic fuzzy T_1 space (IF T_1 space in short) as follows:

Definition 3.3.1 [27]:

Let (X, τ) be an intuitionistic fuzzy topological space, then we say τ is IF T_1 space if and only if for any x, y two distinct elements in X, there exists $U,V \in \tau$ such that $U(x) = V(y) = \underline{1} = <1,0>$, and

U(y) =V(x) = $\underline{0}$ = <0,1>.

Another form of the IF T_1 space using intuitionistic fuzzy points and vanishing intuitionistic fuzzy points comes in the following theorem:

Theorem 3.3.2 [27]:

For an intuitionistic fuzzy topological space (X, τ), the following

are equivalent:

(1) (X, τ) is IF T_1

(2) (i) for any two distinct intuitionistic fuzzy points $a_{(\alpha,\beta)}$, $b_{(\lambda,r)}$ in X there exists U,V $\in \tau$ such that $a_{(\alpha,\beta)} \subseteq U \subseteq b^c_{(\lambda,r)}$ and $b_{(\lambda,r)} \subseteq V \subseteq a^c_{(\alpha,\beta)}$

(ii) for any two distinct vanishing intuitionistic fuzzy points $a_{(\beta)}$, $b_{(r)}$ in X there exists U,V $\in \tau$ such that $a_{(\beta)} \subseteq U \subseteq b^c_{(r)}$ and $b_{(r)} \subseteq V \subseteq a^c_{(\beta)}$.

proof:

(1) ⇒ (2) suppose that (X,τ) is a IF T_1 space,

For (i): let $a_{(\alpha,\beta)}$, $b_{(\lambda,r)}$ be two distinct intuitionistic fuzzy points

Since a ≠b, then by (1) there exists U,V $\in \tau$ such that

U(a) =V(b) = $\underline{1}$ = <1,0>, and U(b) =V(a) = $\underline{0}$ = <0,1>.

That is, $U_1(a) = U_2(b) = V_1(b) = V_2(a) = 1$ and $U_2(a) = U_1(b) = V_2(b) = V_1(a) = 0$

Since $U_1(a)$=1 then $\alpha \leq U_1(a)$ and since $U_2(a)$=0 then $\beta \geq U_2(a)$

Therefore $a_{(\alpha,\beta)} \subseteq U$.

Also since $U_2(b)$=1 then $\alpha \leq U_2(b)$ and since $U_1(b)$=0 then $r \geq U_1(b)$

Therefore, $b_{(\lambda,r)} \subseteq \langle U_2, U_1 \rangle = U^c$, that is to say $U \subseteq b^c_{(\lambda,r)}$, therefore

$a_{(\alpha,\beta)} \subseteq U \subseteq b^c_{(\lambda,r)}$.

Similarly, we prove $b_{(\lambda,r)} \subseteq V \subseteq a^c_{(\alpha,\beta)}$.

For (ii), let let $a_{(\beta)}$, $b_{(r)}$ be two distinct vanishing intuitionistic fuzzy points,since $a \neq b$, then by (1) there exists $U,V \in \tau$ such that

$U(a) = V(b) = \underline{1} = \langle 1,0 \rangle$, and $U(b) = V(a) = \underline{0} = \langle 0,1 \rangle$.

That is, $U_1(a) = U_2(b) = V_1(b) = V_2(a) = 1$ and $U_2(a) = U_1(b) = V_2(b) = V_1(a) = 0$

Since for t=a; $(a^c_{1-\beta})(t) = \beta \geq U_2(a) = 0$, and for $t \neq a$;

$(a^c_{1-\beta})(t) = \beta \geq U_2(a) = 1 \geq U_2(t)$.

Hence,for all t in X, $(a^c_{1-\beta})(t) \geq U_2(t)$.But also, $0 \leq U_1(t)$ for all t in X

Therefore, $a_{(\beta)} \subseteq U$.

Now, since for t=b; $(b^c_{1-r})(t) = r \geq U_1(b) = 0$, and

for $t \neq b$; $(b^c_{1-r})(t) = 1 \geq U_1(t)$, so for all t in X, $(b^c_{r})(t) \geq U_1(t)$.

But also, $0 \leq U_2(t)$ for all t in X therefore $U \subseteq b_r^c$ hence,

$a_{(\beta)} \subseteq U \subseteq b_r^c$, similarly $b_{(r)} \subseteq V \subseteq a_\beta^c$,

(2)⇒ (1) Suppose (i) and (ii) hold,

Let x and y be any two distinct elements in X,consider $a_{(1,0)}$,$b_{(1,0)}$.

By(2) there exist U,V $\in \tau$ such that $a_{(1,0)} \subseteq U \subseteq b_{(1,0)}^c$ and

$b_{(1,0)} \subseteq V \subseteq a_{(1,0)}^c$.

Now, $a_{(1,0)} \subseteq U = <U_1,U_2>$ implies that $U_1(a) \geq 1$ i.e. $U_1(a) = 1$

also, $0 \geq U_2(a)$ which means $U_2(a) = 0$, therefore $U(a) = \underline{1}$.

Similarly $b_{(1,0)} \subseteq V$ implies that $V(b) = \underline{1}$.

Now, $U \subseteq b_{(1,0)}^c \Rightarrow b_{(1,0)} \subseteq U^c = < U_2, U_1 >$.

$1 \leq U_2(b) \Rightarrow U_2(b)=1$ and $0 \geq U_1(b) \Rightarrow U_1(b)=0$ and hence, $U(b) = \underline{0}$.

Similarly $V \subseteq a_{(1,0)}^c \Rightarrow V(a) = 0$, this completes the proof of the theorem.

The definition of intuitionistic fuzzy T_2 space (IF T_2 in short) is by adding a new condition to the IF T_1 definition, namely $U \subseteq V^c$ as in the following definition:

Definition 3.3.3 [27]:

Let (X,τ) be an intuitionistic fuzzy topological space, we say the topology τ is IF T_2 space if and only if for any two distinct elements x,y in X there exists U,V $\in \tau$ such that $U(x) = V(y) = \underline{1} = <1,0>$, U(y) =V(x) = $\underline{0}$ = <0,1> and U ⊆ V^c.

Theorem 3.3.4[27]:

Let (X, τ) be an intuitionistic fuzzy topological space. If the topology is IF T_2 then the topology is IF T_1

Proof: obvious.

The following theorem is parallel to theorem [3.3.2]

Theorem 3.3.5[27]:

For an intuitionistic fuzzy topological space (X, τ), the following are equivalent:

(1) (X,τ) is IF T_2

(2) (i) for any two distinct intuitionistic fuzzy points $a_{(\alpha,\beta)}$, $b_{(\lambda,r)}$ in X there exists U,V ∈ τ such that $a_{(\alpha,\beta)}$ ⊆U ⊆ $b^c_{(\lambda,r)}$, $b_{(\lambda,r)}$⊆ V ⊆ $a^c_{(\alpha,\beta)}$ and U ⊆ V^c.

(ii) for any two distinct vanishing intuitionistic fuzzy points

$a_{(\beta)}$, $b_{(r)}$ in X there exists U,V ∈ τ such that $a_{(\beta)}$ ⊆U ⊆ $b^c_{(r)}$,

$b_{(r)}$⊆ V ⊆ $a^c_{(\beta)}$ and U ⊆ V^c.

proof: similar to the proof of theorem 3.3.2

Chapter Four

Fuzzy Connectedness and Fuzzy Compactness

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4.1 Fuzzy Connected Spaces

Looking back to different equivalent definitions of connectedness in classical topological spaces, one of them was chosen by most of researches to be the extended definition of connectedness in the fuzzy setting.

Definition 4.1.1:

(X, τ) is fuzzy connected if it has no proper fuzzy clopen subset, that is there exist no A fuzzy subset of X such that $A \neq X$, $A \neq \Phi$, and A is both open and closed

The following are two examples of two fuzzy topological spaces on a set X where one of them is connected, while the other one is not.

Example 4.1.2:

consider X= {a, b, c},

let $\tau_1 = \{ \bar{0}, \bar{1}, \{a_{0.3}, b_{0.8}, c_{0.1}\}\}$.

and let $\tau_2 = \{ \bar{0}, \bar{1}, \{a_{0.4}, b_{0.3}, c_{0.6}\}, \{a_{0.6}, b_{0.7}, c_{0.4}\}, \{a_{0.6}, b_{0.7}, c_{0.6}\},$

$\{a_{0.4}, b_{0.3}, c_{0.4}\} \}$.

Then it is clear that τ_1 is connected, but τ_2 is not connected.

Concerning the other definition of connectedness in regular topological spaces which defines a space X to be connected if it can not be written as a union of two non-empty disjoint open sets, the extended definition in fuzzy setting will not be equivalent to the one we adopted,this is explained in the following:

Lemma 4.1.3[11]:

Let A, B be two proper fuzzy subsets of X such that $A \vee B = \bar{1}$ and $A \wedge B = \bar{0}$, then A and B are crisp subsets of X, $A=B^c$ and $B=A^c$.

Proof:

For every $x \in X$, Since $A \vee B = \bar{1}$ then max { A(x), B(x) } = 1 also, since $A \wedge B = \bar{0}$ then min { A(x), B(x) } = 0. Therefore, either A(x) = 0 or 1 , and B(x) = 0 or 1 which means A and B are crisp subsets of X, Moreover, if A(x)=1 then B(x) must be 0 and if A(x)=0 then B(x) must be 1,therefore $A=B^c$ and $B=A^c$.

Theorem 4.1.4[11]:

Let (X,τ) be a fuzzy topological space, if X is connected then X can not be written as a union of two non-empty disjoint fuzzy open subsets of X.

proof:

Assume X is connected. By a contradiction; assume X can be written as a $\bar{1} = A \vee B$ where A and B are non-empty disjoint fuzzy open subsets of X. Since $A \vee B = \bar{1}$ and $A \wedge B = \bar{0}$ then (by lemma 4.1.3) $A= B^c$. Now, since B is open, A is closed. But A is fuzzy open,therefore, A is a non-empty fuzzy clopen proper subset of X which means X is not connected (a contradiction), this completes the proof.

For the other way around, the last theorem is not true.

That is, if X cannot be written as a union of two non-empty disjoint fuzzy open subsets of X with union equal to $\bar{1}$,doesn't imply that X is connected, as the following example shows:

Example 4.1.5:

let X= {a, b} and $\tau = \{\bar{0}, \bar{1}, \{ a_{0.2}, b_{0.3} \}, \{a_{0.8}, b_{0.7} \} \}$.

Here, X could not be written as AVB, where A and B are non-empty disjoint fuzzy open subsets of X,but,still X is disconnected since

A $=\{ a_{0.2}, b_{0.3} \}$ is both open and closed in τ.

In [11] Fatteh and Bassan, modified the condition of connectedness as in the following theorem:

Theorem 4.1.6 [11]:

X is fuzzy connected if and only if there doesn't exist non-empty fuzzy open subsets A and B of X such that: A(x) + B(x) = 1 for every x in X.

proof:

Assume X is fuzzy connected. By contradiction, let A, B be two non-empty fuzzy open subsets of X such that: A(x) + B(x) =1 for every x in X, then $A(x) = 1 – B(x) = B^c(x)$ therefore, $A = B^c$, but B is fuzzy open then A is fuzzy closed, but A is also fuzzy open which means A is fuzzy clopen non-empty subset of X.Hence, X in disconnected (which is a contradiction).

Conversely, by contradiction, assume X is not fuzzy connected then there exist A non empty fuzzy subset of X which is both fuzzy open and fuzzy closed. Take $B = A^c$ then B is fuzzy open and $A(x) + B(x) = A(x)+ A^c (x) = A(x) + (1 – A(x)) = 1$ (which contradicts the assumption).

One of the important differences between connectedness in the regular and fuzzy topological spaces is the property involving product spaces. In the regular topological space, the product of connected spaces is connected,but it is not the case in the fuzzy topological spaces. The following example explains that:

Example 4.1.7:

Let X,Y be connected fuzzy topological spaces,then their product may not be fuzzy connected.

Let X = Y = { a,b,c },

$A=\{a_{0.3},b_{0.9}, c_{0.8}\}$, $B=\{a_{0.7},b_{0.1}, c_{0.2}\}$.

Let $\tau_x = \{0,1,A\}$, $\tau_y=\{0,1,B\}$, then $\tau = \tau_x \times \tau_y$ is not connected

Since $(Ax1)^c = 1 x A^c$, and Ax1 is a proper clopen subset of τ.

Remark:

Let X be a fuzzy topological space, a subset A of X is a fuzzy connected subset if it is fuzzy connected as a fuzzy subspace of X. The same for

A<Y<X, that is, A is a subset of a subspace Y of X,

A is a fuzzy connected subset of X if it is fuzzy connected subset of the fuzzy subspace Y.

Definition 4.1.8 [11]:

Let (X,τ) be a fuzzy topological space,and let A,B be two fuzzy sets, then A,B are said to be separated if and only if Cl(A) V B ≤1 and A V Cl(B) ≤ 1.

The following theorem characterizes connectedness using a condition that is not an extension to any condition in non fuzzy settings:

Theorem 4.1.9 [11]:

Let (X,τ) be a fuzzy topological space. X is connected if and only if there are no non-empty fuzzy subsets A, B of X such that for every x in X:

$$\left.\begin{array}{l} A(x) + B(x) = 1 \\ \bar{A}(x) + B(x) = 1 \\ A(x) + \bar{B}(x) = 1 \end{array}\right\} (1)$$

Proof:

Assume X is connected. By contradiction, let A and B be non-empty fuzzy subsets of X satisfying (1) then:

$B(x) = 1 – A(x) = A^c(x) \Rightarrow B = A^c$

$B(x) = 1 – \bar{A}(x)= \bar{A}^c(x) \Rightarrow B = \bar{A}^c$

$\bar{B}(x) = 1 – A(x) \Rightarrow \bar{B} = A^c$

Since $B = \bar{A}^c$ then B is open, and since $A^c = \bar{B}$ then A^c is closed, but $B = A^c$,therefore B is closed. Hence B is clopen in X (which is a contradiction).

Conversely, by contradiction; assume X is not connected so there exist a non-empty fuzzy clopen proper subset of X, call it D.

Let $C = D^c$, and hence C is clopen.

Now, $D(x) + C(x) = D(x) + (1 – D(x)) = 1$

$\bar{D}(x) + C(x) = D(x) + C(x) =1$ and $D(x) + \bar{C}(x) = D(x) + C(x) =1$

Which is a contradiction.

4.2 Fuzzy Compact Spaces

The concept of compactness is one of the most important concepts in general topology, the notion of fuzzy compactness was first introduced by Chang in terms of open cover.But unfortunately this definition failed to conclude that the product of fuzzy compact sets is fuzzy compact. Many authors were motivated to define new forms of compactness. [see 12, 16, 18].

Definition 4.2.1:

A cover for a fuzzy topological space (X, τ) is a family of members $\{B_\alpha : \alpha \in \Lambda\}$ such that: $VB_\alpha = \bar{1}_x$,that is, Sup $\{ B_\alpha (x):x \in X \} = 1$, and for any fuzzy subset A of X, $\{B_\alpha : \alpha \in \Lambda\}$ is a cover means: $VB_\alpha \geq A$.

A cover is called a fuzzy open cover if each member is a fuzzy open set. A is a subcover of $\{B_\alpha : \alpha \in \Lambda\}$ is a subfamily of $\{B_\alpha : \alpha \in \Lambda\}$ which is also a cover of A.

Now, we define fuzzy compactness parallel to the definition we use in non-fuzzy topological spaces.

Definition 4.2.2:[6]

Let (X,τ) be a fuzzy topological space and let A be a fuzzy subset of X, we say A is a fuzzy compact set if every fuzzy open cover of A has a finite subcover.

Under this definition:

The indiscrete fuzzy topological space is fuzzy compact. Because the only cover for X is $\{ \bar{1}, \bar{0}\}$ which is itself a finite subcover.

Also, in the case of the fuzzy topology τ, where τ is finite, the topological space (X, τ) is compact.

Compactness can be identified using the finite intersection property of fuzzy closed sets as follows:

A family of fuzzy subsets of X { Fα:α∈Λ } has the finite intersection property; as Chang defined,if the intersection of any finite subfamily is not empty.

Theorem 4.2.3 [6]:

A fuzzy topological space (X, τ) is a fuzzy compact if and only if for every collection {A_i: i∈ I } of fuzzy closed sets of X having the finite intersection property, $\Lambda A_i \neq 0$.

Proof:

Let { A_i: i∈I } be a collection of fuzzy closed sets of X with the finite intersection property,suppose that $\wedge A_i = 0$ then $V A_i^c = 1$.

Since X is fuzzy compact, then there exists $i_1, i_2, \ldots, i_n$ such that $V A_{ij}^c = 1$, then $\Lambda A_{ij} = 0$,which gives a contradiction, therefore $\wedge A_{ij} \neq 0$

Conversely;

Let {A_i: i ∈I } be a fuzzy open cover of X. Suppose that for every finite $i_1, i_2, \ldots, i_n$, we have $V A_{i_j} \neq 1$ then $\wedge A_{i_j}{}^c \neq 0$.

Hence { $A_j{}^c \neq 0$ } satisfies the finite intersection property then from the hypothesis we have $\Lambda A_i{}^c \neq 0$.which implies $\wedge A_{i_j}{}^c \neq 1$.

But this contradicts that { A_i: i ∈ I} is a fuzzy open cover of X. Thus X is a fuzzy compact.

We know in non-fuzzy topological spaces that any closed subset of compact space is compact. This property is also valid throughout the fuzzy topological spaces. The following theorem shows that:

Theorem 4.2.4 [6]:

A fuzzy closed subset of a fuzzy compact space is fuzzy compact.

Proof:

Let A be a fuzzy closed subset of a fuzzy compact space X, and let $\{ B_i, i \in I\}$ be any family of fuzzy closed in A with finite intersection property, since A is fuzzy closed in X, then B_i are also fuzzy closed in X, Since X is fuzzy compact, then by previous theorem $\Lambda B_i \neq 0$.

Therefore, A is fuzzy compact.

The fuzzy continuous image of a fuzzy compact set is fuzzy compact, as the following theorem shows:

Theorem 4.2.5[6]:

Let (X,τ_1) and (Y,τ_2) be two fuzzy topological spaces, and let

$f:X\rightarrow Y$ be an onto fuzzy continuous function. Then

X is fuzzy compact $\Rightarrow$ Y is fuzzy compact.

Proof:

Let $\{ B_\alpha \}$ be a family of open sets in Y that covers Y, i.e. $V_\alpha B_\alpha = \overline{1}$.

For each x ∈ X, $V_\alpha f^{-1}(B_\alpha)(x) = V_\alpha B_\alpha (f(x))= 1$,therefore { $f^{-1}(B_\alpha)$ } forms an open cover for X.

But, since X is fuzzy compact, X has a finite sub-cover

$f^{-1}(B_{\alpha_1})$,$f^{-1}(B_{\alpha_1})$,…, $f^{-1}(B_{\alpha_n})$ for X i.e. $V_{i=1}^{n}$ $(f^{-1}(B_{\alpha_i})(x)) = 1$.

Now, since f is onto, $f(f^{-1}(B_{\alpha_i})) = B_{\alpha_i}$and for every y ∈ Y, $(V_{i=1}^{n} B_{\alpha_i})(y) = V_{i=1}^{n} f(f^{-1}(B_{\alpha_i}))(y)$ so

$(V_{i=1}^{n} B_{\alpha_i})(y) = f(V_{i=1}^{n} (f^{-1}(B_{\alpha_i}))(y) = f(1) = 1$.

Therefore, Y is fuzzy compact.

Alexander subbase theorem characterized fuzzy compactness using subbases as in the following:

For any fuzzy topological space (X, τ) and for any subbase S of τ, X is fuzzy compact if and only if every cover of X by members of S has a finite subcover.

We will now prove the Tychonoff property, called Goguen theorem which states that the product of a finite number of fuzzy compact topological spaces is fuzzy compact.

Goguen theorem 4.2.6 [14]:

Let (X_i,τ_i) be a family of fuzzy compact topological spaces

where i = 1, 2, …, n, then $(\prod_{i=1}^{n} X_i, \prod_{i=1}^{n} \tau_i)$ is fuzzy compact, where $\prod_{i=1}^{n} \tau_i$ is the topology generated by the subbase S, where

S = { P_i^{-1}(B_i): $B_i \in \tau_i$, i= 1, 2, …, n } and P_i is the projection

from $\prod_{i=1}^{n} X_i$ to X_i.

proof:

let X = $\prod_{i=1}^{n} X_i$ and $\tau = \prod_{i=1}^{n} \tau_i$

Let S = { P_i^{-1} (B_i): $B_i \in \tau_i$, i= 1, 2, …, n } be a subbase for τ and

let $\mathcal{C}$ be a family of members of S,

Let $\mathcal{C}_i$ = {$B_i \in \tau_i$: such that P_i^{-1} (B_i)$\in \mathcal{C}$ }. Then $\mathcal{C}_i$ is a family of open

fuzzy sets in τ_i that is a cover. But, τ_i is compact, then there exists a finite subcover $B_{i,1}$, $B_{i,2}$, …, $B_{i,k}$ such that: $\bigvee_{j=1}^{k} B_{i\,j} = 1_{X_i}$ and therefore;

$\bigvee_{i=1}^{n}(\bigvee_{j=1}^{k} P_i^{-1}$ ($B_{i\,j}$)) = $\bigvee_{i=1}^{n}(P_i^{-1}(\bigvee_{j=1}^{k}$ ($B_{i\,j}$))) = $\bigvee_{i=1}^{n}(P_i^{-1}(1_{X_i})$

Hence $\bigvee_{i=1}^{n}(\bigvee_{j=1}^{k} P_i^{-1}$ ($B_{i\,j}$)) = $\bigvee_{i=1}^{n}(1_X) = 1_X$.

In the following example, we will show that the infinite product of fuzzy compact topological spaces may not be compact.

Example 4.2.7:

let X_i = N = { 1, 2, 3, … }, for each i = 1, 2, 3, … we define the

fuzzy topological space as follows:

$\tau_1 = \{ \bar{0}, \bar{1} \}$

$\tau_2 = \{\ \bar{0}, \bar{1}, \overline{\tfrac{1}{2}}, \{\ 1_{\frac{1}{2}}\}, \{\ 1_{\frac{1}{2}}, 2_{\frac{1}{2}}\ \}, \{\ 1_{\frac{1}{2}}, 2_{\frac{1}{2}}, 3_{\frac{1}{2}}\ \}, \dots \}$

$\tau_n = \{\ \bar{0}, \bar{1}, \overline{\tfrac{n-1}{n}}, \{\ 1_{\frac{n-1}{n}}\}, \{\ 1_{\frac{n-1}{n}}, 2_{\frac{n-1}{n}}\ \}, \{\ 1_{\frac{n-1}{n}}, 2_{\frac{n-1}{n}}, 3_{\frac{n-1}{n}}\ \}, \dots \}$

Then each τ_n, n = 1, 2, 3,... is compact, because for any open cover for τ_n should contain $\bar{1}$ and therefore has a finite subcover; $\bar{1}$ itself.

To show: $(\prod X_i, \prod \tau_i)$ is not compact,

call $B_{i,n} = P_i^{-1}$ ({ $1_{\frac{n-1}{n}}, 2_{\frac{n-1}{n}}, \dots, n_{\frac{n-1}{n}}$ }), { $B_{i,n}$ } is open in $\prod X_i$

we show V $B_{i,n}$ (x) = 1.

$$B_{i,n}(x) = \begin{cases} \frac{i-1}{i} & if\ x_i \le n \\ 0 & if\ x_i > n \end{cases}$$

therefore, for any ϵ > 0, there exists m ∈N such that 1- ϵ < $\frac{m-1}{m}$

If n ≥ x_m then $B_{i,n}(x_i)$ > 1- ϵ which means Sup { $B_{i,n}(x_i)$ } = 1, that is $\bigvee_{i,n\in N} B_{i,n}$(x) = 1,hence $\{ B_{i,n} \}_{\substack{n=1 \\ i=1}}^{\infty}$ is an open cover for X.

For any finite subfamily B_{i_1,n_1}, B_{i_2,n_2}, …, B_{i_t,n_t}; we can find k such that: for n_i > k, $B_{i_{n_i}}$(x) = 0 where x= (k, k, …, k) and therefore $\mathrm{V}B_{i_{n_0}}$< 1, hence, any finite subfamily is not a cover, and X is not compact

Chapter Five

Fuzzy Continuous Functions

Chapter Five
Fuzzy Continuous Functions

5.1 Fuzzy Functions

In chapter one we have defined a fuzzy function $\bar{f}$ from F(X) to F(Y) as an extension to a function f from a set X to a set Y. The following theorem explores the properties of the fuzzy functions over fuzzy subsets of X and Y

Theorem 5.1.1 [35]:

let f:X → Y be a function and $\bar{f}$:F(X) → F(Y) be the corresponding fuzzy function, then for any fuzzy subsets A and B of X and any fuzzy subsets L and M of Y, we have the following:

1) $\bar{f}$ (A V B) = $\bar{f}$ (A) V $\bar{f}$ (B)

2) $\bar{f}$ (A Λ B) ≤ $\bar{f}$ (A) Λ $\bar{f}$ (B)

3) $(\bar{f})^{-1}$ (L V M) = $(\bar{f})^{-1}$(L) V $(\bar{f})^{-1}$ (M)

4) $(\bar{f})^{-1}$(L Λ M) = $(\bar{f})^{-1}$ (L) Λ $(\bar{f})^{-1}$(M)

proof:

1) Let K = A V B then

$$\bar{f}(K)(y) = \begin{cases} \sup\{ (K)(x): x \epsilon f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{if } f^{-1}(y) = \emptyset \end{cases}$$

now;

if $f^{-1}(y) = \emptyset$ then $\bar{f}$ (A)(y) = 0 and $\bar{f}$ (B)(y) = 0 and therefore,

$(\bar{f}(A) \vee \bar{f}(B))(y) = 0$, also $\bar{f}$ (K)(y) = 0 hence $(\bar{f}(A) \vee \bar{f}(B))(y) = \bar{f}(K)(y)$

and, if $f^{-1}(y) \neq \emptyset$

$\bar{f}(K)(y) = \sup \{ (K)(x): x \in f^{-1}(y) \}$

$= \sup \{ \max \{ A(x), B(x): x \in f^{-1}(y) \}$

$= \max \{ \sup \{ A(x) \}, \sup \{ B(x) \}: x \in f^{-1}(y) \}$

$= \max \{ \bar{f}(A)(y), \bar{f}(B)(y) \} = (\bar{f}(A) \vee \bar{f}(B))(y)$

hence $\bar{f}(A \vee B) = (\bar{f}(A) \vee \bar{f}(B))$

2) if $f^{-1}(y) = \emptyset$ (easy)

if $f^{-1}(y) \neq \emptyset$ then:

$\bar{f}(A \wedge B)(y) = \sup \{ (A \wedge B)(x): x \in f^{-1}(y) \}$

$= \sup \{ \min \{ A(x), B(x) \}: x \in f^{-1}(y) \}$

$\leq \min \{ \sup \{ A(x) \}, \sup \{ B(x) \} \}$

$= (\bar{f}(A) \wedge \bar{f}(B))(y)$

hence $\bar{f}(A \wedge B) \leq (\bar{f}(A) \wedge \bar{f}(B))$

3) $(\bar{f})^{-1}(L \vee M)(x) = L \vee M(f(x))$

$= \max \{ L(f(x)), M(f(x)) \}$

$= \max \{ (\bar{f})^{-1}(L)(x), (\bar{f})^{-1}(M)(x) \}$

$= ((\overline{f}\,)^{-1}(L) \vee (\overline{f}\,)^{-1}(M))(x)$

Hence, $(\overline{f}\,)^{-1}(L \vee M) = ((\overline{f}\,)^{-1}(L) \vee (\overline{f}\,)^{-1}(M))$

4) $((\overline{f}\,)^{-1}(L \wedge M)(x) = L \wedge M\,(f(x))$

$= \min \{ L(f(x)), M(f(x)) \}$

$= \min \{ (\overline{f}\,)^{-1}(L)(x), (\overline{f}\,)^{-1}(M)(x) \}$

$= ((\overline{f}\,)^{-1}(L) \wedge (\overline{f}\,)^{-1}(M))(x).$

Therefore, $(\overline{f}\,)^{-1}(L \wedge M) = ((\overline{f}\,)^{-1}(L) \wedge (\overline{f}\,)^{-1}(M))$

And in general let $\{B_\alpha\}$ be a family of fuzzy subsets of Y, then:

(i) $(\overline{f}\,)^{-1}(\vee B_\alpha) = \vee (\overline{f}\,)^{-1}(B_\alpha)$, and

(ii) $(\overline{f}\,)^{-1} (\wedge B_\alpha) = \wedge (\overline{f}\,)^{-1}(B_\alpha)$

5.2 Fuzzy Continuity

In the following, we will define the continuous fuzzy functions between two fuzzy topological spaces. Also, we will study their properties in both weaker and stronger forms of fuzzy continuity.

Definition 5.2.1:[6]

f: $(X, \tau_X) \to (Y, \tau_Y)$ is said to be fuzzy continuous if and only if the inverse image of any fuzzy open set in Y is a fuzzy open set in X, where τ_X is a fuzzy topology on X and τ_Y is a fuzzy topology on Y.

Considering a fuzzy continuity as a local property, we have the following definition:

Definition 5.2.2:

f: (X, τ_x) → (Y, τ_Y) is fuzzy continuous at any fuzzy point p =x_λ if and only if for every nbd V of $\bar{f}$(p)=f(x)$_\lambda$, there exist U ∊ τ_X such that p ∊ U and f(U) ≤ V.

The above two definitions are related as the following theorem states.

Theorem 5.2.3 [37]:

$\bar{f}$: (X, τ_x) → (Y, τ_Y) is fuzzy continuous if and only if $\bar{f}$ is fuzzy continuous at each fuzzy point p in X.

proof:

let $\bar{f}$: (X, τ_x) → (Y, τ_y) be fuzzy continuous, p= x_λ a fuzzy point

in X,V be a fuzzy nbd of $\bar{f}$(p)=f(x)$_\lambda$ in Y.

There exist V_1 in τ_y such that: $\bar{f}$(p) ∈ V_1 ≤ V.

Since f is fuzzy continuous, U = $(\bar{f})^{-1}(V_1)$ is fuzzy open and contains x_λ

So $\bar{f}(U) = \bar{f}\,(\bar{f}^{-1}(V_1)) \leq V_1 < V$.

Conversely,

let B∈ τ_y, p= x_λ a fuzzy point in f^{-1}(B), f(p)=f(x)$_\lambda$=y_λ

Then $f(p) \in f(f^{-1}(B)) \leq B$.

Now, $\bar{f}(p)(y_\lambda)$ = sup $B(a)$: $a \in \bar{f}^{-1}(y_\lambda) = \bar{f}(B)(y_\lambda)$.

So there exist a fuzzy nbd U of p such that $\bar{f}(U) \leq B$.

So $p \in U \leq \bar{f}^{-1}(B)$, and there exist U_1 in τ_x such that $p \in U_1 \leq U$

i.e. $p \in U_1 \leq \bar{f}^{-1}(B)$

taking the union of all p implies that

$\bar{f}^{-1}(B) = V\{p$: p is a fuzzy point in $\bar{f}^{-1}(B)\} \leq V\{U_1\} \leq \bar{f}^{-1}(B)$

so $\bar{f}^{-1}(B) = V\{U_1\} \in \tau_x$. That is, $\bar{f}^{-1}(B)$ is a fuzzy open set in X.

Hence, $\bar{f}$ is fuzzy continuous.

Remark 5.2.4:

We know that in regular topological spaces, any constant function is a continuous function, whatever the topologies defined on X and Y. But, this is not the case in fuzzy functions on fuzzy topological spaces as shown in the following example.

Example:

Let X= { a, b} with $\tau_x=\{\bar{0}, \bar{1}_x, \{a_{0.2}, b_{0.5}\}\}$

And let Y= { c, d} with $\tau_y=\{\bar{0}, \bar{1}_y, \{c_{0.4}, d_{0.5}\}\}$

Let f: X→ Y be the constant fuzzy function f(X)={ c }.

Looking at $\bar{f}$:F(X) → F(Y), take V = { $c_{0.4}$, $d_{0.5}$} then:

$(\bar{f})^{-1}(V) = (\bar{f})^{-1}$ { $c_{0.4}$, $d_{0.5}$} ={ $a_{0.4}$, $b_{0.4}$} which is not open in X.

Therefore, this constant function is not continuous. Probably, that was the reason Lowen has suggested another definition for fuzzy topological spaces where he replaced the first condition of Chang's definition (namely $\bar{0}$, $\bar{1} \in \tau$) by $\bar{r} \in \tau$ for r ∈[0,1] where $\bar{r}$ ={ x_r: for every x in X } and named this fuzzy topology by L-fuzzy topology corresponding to C-fuzzy topology of Chang. And therefore, in L-fuzzy topology, any constant function is continuous.

Theorem 5.2.5:

In L-fuzzy topology, every constant function is a fuzzy continuous function

proof:

Let f: X→ Y be a constant function (i.e. f(x)=c for all x in X),

and let V be any fuzzy open set in Y.

if V(c) = 0 then $(\bar{f})^{-1} = \bar{0}$ which is a fuzzy open in X,

and if V(c) = r ≠ 0 then $(\bar{f})^{-1}(V) = \bar{r}$ which is again a fuzzy open in X.

Therefore, $\bar{f}$ is fuzzy continuous.

Theorem 5.2.6 [37]:

$\bar{f}$ is a fuzzy continuous function if and only if the inverse image of any fuzzy closed set in Y is a fuzzy closed in X.

proof:

let F be any fuzzy closed set in Y then F^c is fuzzy open set in X

but $\bar{f}$ is fuzzy continuous so $(\bar{f})^{-1}(F^c) = ((\bar{f})^{-1}(F))^c$ is fuzzy open

set in X i.e. $(\bar{f})^{-1}(F)$ is fuzzy closed in X.

conversely, let V be fuzzy open in X then V^c is fuzzy closed in Y

so $(\bar{f})^{-1}(V^c) = ((\bar{f})^{-1}(V))^c$ is fuzzy closed in X, (i.e. $(\bar{f})^{-1}(V)$ is

fuzzy open in X.

Therefore, $\bar{f}$ is fuzzy continuous function.

5.3 Other Types of Fuzzy Continuity

In standard topological spaces, the concepts of regular open sets, regular closed sets and almost continuous functions were defined and studied. In fuzzy setting, parallel definitions are presented as follows:

Fuzzy almost continuity

Definition 5.3.1 [cf 26]:

Let A be a fuzzy subset of X, we say A is fuzzy regular open set if int(Cl (A)) = A,

and A is fuzzy regular closed set if cl(int (A)) = A.

Definition 5.3.2:

Let f:X → Y be a function, we say $\bar{f}$:F(X) → F(Y) is fuzzy almost continuous if for every V is fuzzy regular open in Y; $\bar{f}^{-1}$(V) is fuzzy open in X.

Concerning the above definition, we have some properties presented in the following remark:

remark 5.3.3:

(1) if A is fuzzy regular open then A is fuzzy open in X. Because A is fuzzy regular open, so A = int (cl (A)) which is the interior of some fuzzy set, hence it is open.

(2) Similarly, if A is a fuzzy regular closed in X then A is fuzzy Closed set in X.

(3) A is regular fuzzy open if and only if A^c is regular fuzzy closed

Proof:

A is regular fuzzy open means A = int (Cl(A)).

So, $A^c = (\text{int}\ (\text{Cl}(A)))^c = 1 - \text{int}\ (\text{Cl}(A)) = \text{Cl}\ (1 - \text{Cl}(A))$,

hence, $A^c = \text{Cl}(\text{int}\ (1- A) = \text{Cl}(\text{int}\ (A^c))$

Therefore, A^c is a regular fuzzy closed.

(4) Every fuzzy continuous is almost fuzzy continuous. Because for any V fuzzy regular open in Y, V is fuzzy open in Y. But $\bar{f}$ is fuzzy continuous then $(\bar{f})^{-1}(V)$ is fuzzy open in X which concludes that $\bar{f}$ is almost fuzzy continuous.

Theorem 5.3.4 [26]:

Let $\bar{f}:(X,\tau_x) \rightarrow (Y,\tau_x)$ be a fuzzy function then the following are equivalent:

1) $\bar{f}$ is a fuzzy almost continuous function

2) for every B regular fuzzy closed in Y; $\bar{f}^{-1}(V)$ is fuzzy closed in X.

proof:

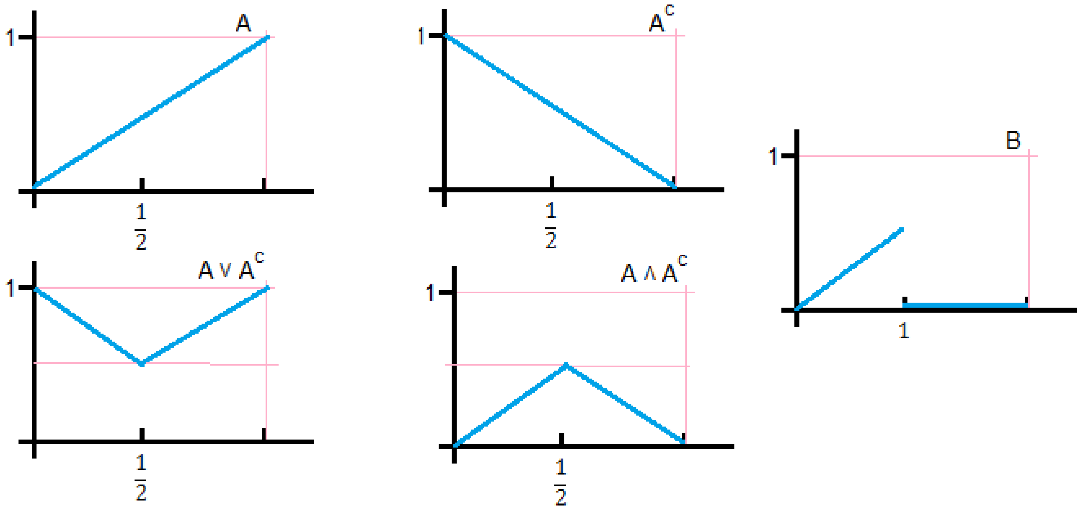
using the facts: $(\bar{f}^{-1})(B^c) = ((\bar{f}^{-1})(B))^c$ for any B:fuzzy subset of X, and that the complement of a regular fuzzy open is fuzzy regular closed, the two statements are equivalent.

Example 5.3.5:(fuzzy almost continuity doesn't imply fuzzy continuity)

Consider X = Y = [0,1],

let τ_X= { $\bar{0}$, $\bar{1}$, A, A^c, A V A^c, A $\wedge$ A^c }

and τ_Y = { $\bar{0}$, $\bar{1}$, A, A^c, A V A^c, A $\wedge$ A^c, B } where



We have, A,, A^c, A V A^c, A $\wedge$ A^c are both fuzzy open and fuzzy closed in both τ_X and $\tau_{Y,}$ and therefore, they are both regular fuzzy open and regular fuzzy closed in both τ_X and $\tau_{Y.}$ But, B is fuzzy open in Y but it is not regular fuzzy open.

Let f: X→ Y, be the identity function f(x) = x for all x ∊ X It is clear that $\overline{f}$ is a fuzzy almost continuous function but it is not fuzzy continuous function because $(\overline{f}^{-1})$(B)=B which is not fuzzy open in X.

Many authors defined different types of fuzzy open and fuzzy closed sets and used them to define and study new types of fuzzy continuous functions. Let us look at some of those types.

Fuzzy $\boldsymbol{\delta}$-continuous

Another type of continuity is called a fuzzy $\boldsymbol{\delta}$-continuity defined as follows:

Definition 5.3.6:

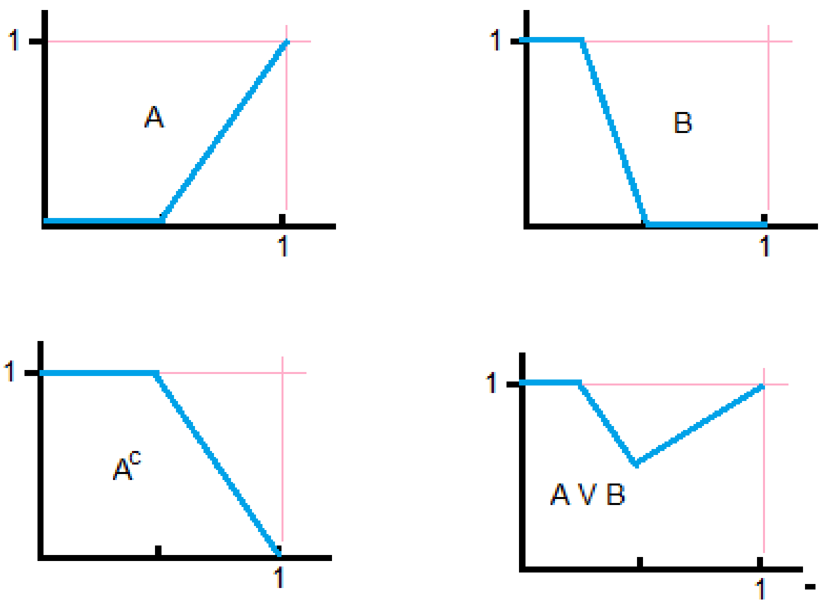
Let $\bar{f}$:be a fuzzy function between the topological spaces (X,τ_x) and (Y,τ_y) then we say $\bar{f}$ is fuzzy $\boldsymbol{\delta}$-continuous if and only if for each fuzzy point p in X, and each fuzzy regular open set B containing $\bar{f}$(p), there exist a fuzzy regular open set A containing p such that: $\bar{f}(A) \leq B$.

The following is an example of a fuzzy function that is fuzzy continuous but not fuzzy $\boldsymbol{\delta}$-continuous:

Example 5.3.7:

Let X = Y = [0,1] and let τ_x and τ_y be two fuzzy topological spaces on X and Y respectively where:

τ_X = {$\bar{0}$, $\bar{1}$, A^c} and τ_y= {$\bar{0}$, $\bar{1}$,A, B,AVB} where the members of these topologies are shown in the following graphs:



Now, define the function f:X→Y by: f(x) = $\frac{x}{2}$ for all x in X

We can see that $(\bar{f})^{-1}(\bar{0}_Y) = \bar{0}_X$ and $(\bar{f})^{-1}(\bar{1}_Y) = \bar{1}_X$

$(\bar{f})^{-1}$ (A) = $\bar{0}_X$, $(\bar{f})^{-1}$(B) = A^c and $(\bar{f})^{-1}$(A V B) = A^c.

Therefore, $\bar{f}$ is fuzzy continuous.

Now, let p be a fuzzy point in X such that f(p)∈A or f(p)∈B.

There is no fuzzy regular open U in X containing p such that

$\bar{f}(U) \leq A$ or $\bar{f}(U) \leq B$.

Hence, f is not $\boldsymbol{\delta}$-continuous

Fuzzy Precontinuous:

another type of continuity(fuzzy pre-continuous)

Definition 5.3.8:

Let (X,τ) be a fuzzy topological space, we say a fuzzy subset A of X is fuzzy preopen if $A \leq int(cl\ (A))$

We say a fuzzy set B is fuzzy preclosed if and only of B^c is fuzzy preopen.

Using preopen fuzzy sets we define the following;

Definition 5.3.9 [33]:

A function $\bar{f} : (X,\tau_x) \rightarrow (X,\tau_Y)$ is called fuzzy precontinuous if and only if for each B fuzzy open in Y, $(\bar{f})^{-1}$(B) is fuzzy preopen in X.

Definition 5.3.10 [33]:

A function $\overline{f}$: (X,τ_x) → (X,τ_Y) is called a fuzzy slightly precontinuous function if and only if for every B fuzzy clopen set in Y, $(\overline{f})^{-1}(B)$ is fuzzy preopen in X

Ekici in his paper [10] proved a relationship between fuzzy

slightly precontinuity and fuzzy precontinuous:

Theorem 5.3.11 [10]:

Let Y have a base consisting of fuzzy clopen sets. If $\overline{f}$: F(X) → F(Y) is fuzzy slightly precontinuous, then $\overline{f}$ is fuzzy precontinuous.

Generalized Fuzzy Continuity:

Definition 5.3.12:[25]

A fuzzy set A is a generalized fuzzy closed (GFC) if for A ≤ U then cl(A) ≤ U for any U fuzzy open and we say a set B is a generalized fuzzy open (GFO) if 1 – B is GFC.

Definition 5.3.13 [25]

Also, $\overline{f}$ is a generalized fuzzy continuous if the inverse image of a fuzzy open set in Y is GFO in X.

Also Ramish [19] defined a new class of open and closed fuzzy sets on intuitionistic fuzzy topological spaces as follows:

Definition 5.3.14 [19]:

An intuitionistic fuzzy set A is intuitionistic fuzzy regular weakly generalized closed set (IFRWGCS): if $A \leq U$ then $Cl(int(A)) \leq U$ for every U an intuitionistic open fuzzy set.

Definition 5.3.15 [19]:

An intuitionistic fuzzy function $\bar{f}$: IF(X) → IF(Y) is intuitionistic almost fuzzy continuous if the inverse image of an intuitionistic open fuzzy subset of Y is IFRWGCS in X.

Discussion and conclusion

Through this study it was found that many properties of topological spaces in non fuzzy setting were extended to topological spaces in fuzzy settings. However, some other properties were not extended, which motivated the researchers to put down new definitions to conclude parallel theorems.

Since there have been different definitions for the same property, this causes researches and studies to be scattered, there have to be a unification of definitions of different properties that will orient the research by all interested people to be in one direction, and all efforts would be strengthened.

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جامعة النجاح الوطنية
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توسعة الخصائص التبولوجية للفراغات التبولوجية الضبابية

إعداد
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قدمت هذه الأطروحة استكمالاً لمتطلبات الحصول على درجة الماجستير في الرياضيات بكلية الدراسات العليا في جامعة النجاح الوطنية في نابلس، فلسطين.
2014م

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الملخص

في هذه الرسالة قمنا بالتحري عن الخصائص التبولوجية للفراغات التبولوجيه الضبابية وربطها بتلك الخصائص للفراغات التبولوجية الكلاسيكية.

أيضا، تم عرض المجموعات و الاقترانات و العلاقات الضبابية مع خصائصها. ثم تم تقديم أنواع مختلفة من الفضاءات التبولوجية الضبابية حسب مفهوم تشانغ و لون، وكذلك الفراغات التبولوجية الضبابية الحدسية. لقد تم اثبات ان العديد من الخصائص التبولوجية هي توسعة لتلك الخصائص في البيئة الغير ضبابية، بينما تم عرض أمثلة للخصائص التي تتوافق بين البيئتين الضبابية و غير الضبابية مثل مغلق حاصل الضرب لا يساوي حاصل ضرب المغلق.

وكذلك تم التحري عن المسارات المختلفة لفرضيات الانفصال باستخدام الجوار Q و كذلك النقاط الضبابية. وقد تبين ان معظم هذه الخصائص ليست التوسعة الطبيعية.

وأيضا، تم دراسة الترابط الضبابي و التراص الضبابي، وظهر أن خاصية الضرب لعدد لانهائي من الفراغات الضبابية المتراصة ليست بالضرورة متراصة.

وأخيرا تم تقديم مفاهيم الاتصال الضبابي و شبه المتصل الضبابي وأنواع أخرى من الاتصالات الضبابية واثباتات للعلاقات الرابطة بينها.