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Fuzzy Fredholm Integral Equation of the Second Kind

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Dedication

I like to dedicate this thesis to my father, my aunt Nawal, and my friends, who stood by me, encouraged me, and believed in me.

Acknowledgment

First of all, I would like to express my most sincere thanks and appreciation to my supervisor Prof. Dr. Naji Qatanani for introducing me to this subject and for his support, advice and excellent guidance. My thanks also to my external examiner Dr. Iyad Suwwan and to my internal examiner Dr. Mohammad Al-Amleh for their useful and valuable comments. My thanks also to faculty members of the department of mathematics at An-Najah National University.

الاقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان:

Fuzzy Fredholm Integral Equation of the Second Kind

معادلة فريدهولم التكاملية الضبابية من النوع الثاني

أقر بأن ما اشتملت عليه هذه الرسالة إنما هي نتاج جهدي الخاص، باستثناء ما تمت الإشارة اليه حيثما ورد، و أن هذه الرسالة ككل، أو أي جزء منها لم يقدم من قبل لنيل أية درجة علمية أو بحثي لدى أية مؤسسة تعليمية أو بحثية أخرى.

Declaration

The work provided in this thesis, unless otherwise referenced, is the research's own work, and has not been submitted elsewhere for any other degree or qualification.

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Fuzzy Fredholm Integral Equation of the Second Kind

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Abstract

Fuzzy Fredholm integral equations of the second kind have received considerable attention due to the importance of these types of equations in studies associated with applications in mathematical physics and fuzzy financial and economic systems.

After addressing the basic concepts of fuzzy integral equations, we have investigated the analytical and the numerical aspects of the fuzzy Fredholm integral equation of the second kind. The analytical methods include: Fuzzy Laplace transform method, Homotopy analysis method (HAM), Adomain decomposition method (ADM) and Fuzzy differential transformation method (FDTM).

For the numerical treatment of the fuzzy integral equation of the second kind, we have employed the Taylor expansion method and the trapezoidal method. Some numerical test cases are included. A comparison between the analytical and the numerical methods has been presented. Numerical results have shown to be in a closed agreement with the analytical ones.

Introduction

Fuzzy integral equations have attracted the attention of many scientists and researchers in recent years, due to their importance in applications, such as Fuzzy control, Fuzzy financial, approximate reasoning and economic system, etc.

The concept of integration of fuzzy functions was first introduced by Dubois and Prade [29]. Then alternative approaches were later suggested by Goetschel and Voxman [40], Kaleva [48], Matloka [55], Nanda [59], and others. While Goetschel and Voxman [40], and later Matloka [55], preferred a Riemann integral type approach, Kaleva [48], choose to define the integral of fuzzy function using the Lebesgue type concept for integration. Park et al. [46], have considered the existence of solution of fuzzy integral equation in Banach space.

Fuzzy Fredholm integral equation of the second kind is one of the main fuzzy equations addressed by many researchers. Wu and Ma [28] investigated the Fuzzy Fredholm integral equation of the second kind, which is one of the first applications of fuzzy integration.

Since it is difficult to solve Fuzzy Fredholm integral equations analyticaly, numerical methods have been proposed. For instance, Maleknejad solved the first kind Fredholm integral equation by using the sinc function [9]. Parandin and Araghi established a method to approximate the solution using finite and divided differences methods [18].

Jafarzadeh solved linear fuzzy Fredholm integral equation with Upper – bound on error by Splinder's Interpolation [64]. Altaie used Bernstein piecewise polynomial [16]. Parandin and Araghi proposed the approximate solution by using an iterative interpolation [19]. Lotfi and Mahdiani used Fuzzy Galerkin method with error analysis [54]. Attari and Yazdani studied the application of Homotopy perturbation method [17]. Mirzaee, Paripour and Yari presented direct method using Triangular Functions [57]. Gohary and Gohary found an approximate solution for a system of linear fuzzy Fredholm integral equation of the second kind with two variables which exploit hybrid Legendre and block-pulse functions, and Legendre wavelets [39]. Ziari, Ezzati and Abbasbandy used Fuzzy Haar Wavelet [4]. Ghanbari, Toushmalni and Kamrani Presented a numerical method based on block-pulse functions (BPFs) [37].

In this thesis, some analytical and numerical methods for solving fuzzy Fredholm integral equation of the second kind will be investigated. Using the parametric form of fuzzy numbers, the fuzzy linear Fredholm integral equation of the second kind can be converted to a linear system of Fredholm integral equations of the second kind in the crisp case.

In chapter one of this thesis, we introduce some basic concepts in fuzzy mathematics such as crisp sets, fuzzy sets and fuzzy numbers. In chapter two, we study the existence and uniqueness of the solution of fuzzy Fredholm integral equation of the second kind. Chapter three includes an investigation of some analytical methods used to solve fuzzy Fredholm integral equations of the second kind. These include: Fuzzy Laplace transform method, Homotopy analysis method (HAM), Adomain decomposition method (ADM) and Fuzzy differential transformation method (FDTM). In chapter four, we use two well-known numerical methods, namely: Taylor expansion method and Trapezoidal method to solve fuzzy integral equations. Numerical examples with algorithms are presented in chapter five. Finally, we will draw a comparison between the exact and numerical solutions for some cases.

Chapter One

Mathematical Preliminaries

Chapter One

Mathematical Preliminaries

1.1 Crisp sets

The concept of a set is fundamental in mathematics and it can be described as a collection of objects possibly linked through some properties.

Definition (1.1): Let X be a set and A be a subset of X ($A \subseteq X$). Then a crisp set A is defined as a mapping from element of X to elements of the set $\{0,1\}$.

Definition (1.2) [25]: Let X be a set and A be a subset of X ($A \subseteq X$). Then the characteristic function of the set A in X is defined by:

$$\chi_{A(x)} = \begin{cases} 1 & if x \in A \\ 0 & if x \notin A \end{cases}$$

We represent the crisp sets and their operations using the characteristic function. Let us consider the union $A \cup B = \{x \in X | x \in A \text{ or } x \in B\}$. Its characteristic function is $\chi_{A\cup B(x)} = \max\{\chi_{A(x)}, \chi_{B(x)}\}$.

For the intersection $A \cap B = \{x \in X | x \in A \text{ and } x \in B\}$ the characteristic function is $\chi_{A\cap B(x)} = \min\{\chi_{A(x)}, \chi_{B(x)}\}$.

If we consider the complement of A in X, $A^c = \{x \in X | x \notin A\}$ it has the characteristic function: $\chi_{A^c(x)} = 1 - \chi_{A(x)}$.

1.2 Fuzzy Sets

Definition (1.3) [65]: A fuzzy set A in X is characterized by a membership (characteristic) function $A(x)$ which associates with each point x ∈ X a real number A(x) ∈ [0, 1], with the value of $A(x)$ at x representing the "grade of membership" of x in the fuzzy set A.

Definition (1.4) [25]: Let X be a set, A is a fuzzy subset of X defined as a mapping $A : X \to [0,1]$, where A is called the membership function and the value $A(x)$ is called the membership degree of x to the fuzzy subset A.

The crisp sets represented by their characteristic functions $\chi_A: X \to \{0,1\}$.

Let $A : X \to [0,1]$ if the membership degree of x in the set A is $A(x) = 1$ then it's called full membership, and if $A(x) = 0$ then it's called non-membership. So fuzzy sets are generalizations of crisp sets since other membership degree are allowed.

Definition (1.5) [25]: Let $A : X \to [0,1]$ be a fuzzy set. The level sets of A are defined as the crisp sets:

1) $A_\alpha = \{x \in X | A(x) \geq \alpha\}$, where $0 < \alpha \leq 1$.

2) $A_0 = cl\{x \in X: A(x) > 0\} = \overline{\{x \in X: A(x) > 0\}}$.

3) The strong α-level is $A_{\alpha+} = \{x \in X | A(x) > \alpha\}$.

4) The core of the fuzzy set A is $A_1 = \{x \in X | A(x) \geq 1\}$.

5) The support of the fuzzy set $A = suppA = \{x \in X | A(x) > 0\}$.

Definition (1.6) [33]: A fuzzy set is convex if each of its α-level are convex set, $i.e. A_\alpha = \{x \in X | A(x) \geq \alpha\}$ are convex $\forall \alpha \in (0,1]$.

An alternative definition of convexity: we call A convex if and only if

$A(\lambda x + (1-\lambda)y) \geq \min\{A(x), A(y)\}, \forall x, y \in X, \lambda \in [0,1]$.

Definition (1.7): A fuzzy set is normal if at least one of its elements attains full membership, $i.e. A$ is normal if $\exists x \in X$ such that $A(x) = 1$.

We have basic connectives in fuzzy set theory (inclusion, union, intersection and complement) which are performed on the membership functions since they represent the fuzzy sets.

Definition (1.8) [25]: Let A and B be two fuzzy sets, then:

1) Inclusion**:** we say that the fuzzy set A included in B if

   $A(x) \leq B(x), \forall x \in X$.

2) The intersection of A and B is the fuzzy set C with $C(x) = (A \cap B)(x) = \min\{A(x), B(x)\} = A(x) \wedge B(x), \forall x \in X$.

3) The union of A and B is the fuzzy set C, where

   $C(x) = (A \cup B)(x) = \max\{A(x), B(x)\} = A(x) \vee B(x), \forall x \in X$.

4) equilibrium points $A(x) = A^c(x)$.

5) The complement of A is the fuzzy set B, where

   $B(x) = A^c(x) = 1 - A(x), \forall x \in X$.

6) Difference $(A - B)(x) = (A \cap B^c)(x) = \min\{A(x), 1 - B(x)\}$.

1.3 Fuzzy Numbers

Fuzzy numbers generalize classical real numbers and we can say that a fuzzy number is a fuzzy subset of the real line which has some additional properties. The concept of fuzzy number is vital for fuzzy analysis, fuzzy differential equations and fuzzy integral equations, and a very useful tool in several applications of fuzzy sets.

Definition (1.9) [49]: A fuzzy number is a fuzzy set of the real line

$u : \mathbb{R} \to [0,1]$ satisfying the following properties:

1) u is normal, $i.e.$ there exist as $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$.

2) u is fuzzy convex $(i.e. u(\lambda x + (1-\lambda)y)\lambda \geq \min\{u(x), u(y)\}$,

   $\forall x, y \in \mathbb{R}, \lambda \in [0,1])$.

3) u is upper semi-continuous on $\mathbb{R}$ $(i.e. \forall \epsilon > 0, \exists \delta > 0$ such that

   $u(x) - u(x_0) < \epsilon, |x - x_0| < \delta)$.

4) u is compactly supported, that is $\overline{\{x \in \mathbb{R}: u(x) > 0\}}$ is compact.

The set of all fuzzy real numbers is denoted by E^1. This fuzzy number space as shown in [28], can be embedded into the Banach space.

Definition (1.10) [44]: Let the membership function $A(x)$ of the fuzzy number $A = (a_1, a_2, a_3, a_4)$ has the form:

$$A(x) = \begin{cases} 0 & if \quad x < a_1 \\ f(x) & if \quad a_1 \le x \le a_2 \\ 1 & if \quad a_2 \le x \le a_3 \\ g(x) & if \quad a_3 \le x \le a_4 \\ 0 & if \quad x > a_4 \end{cases} \qquad (1.1)$$

where f is an increasing function and is called the left side, while g is a decreasing function and is called the right side. If f and g are continuous functions then (1.1) is called Trapezoidal fuzzy number.

a_1 *and* a_4 called the outer borders, a_2 *and* a_3 are inner borders of the fuzzy number A. If $a_2 = a_3$ then (1.1) is called triangular fuzzy number.

An alternative definition of fuzzy number is as follows:

Definition (1.11) [45]: A fuzzy number u in parametric form is a pair $\left(\underline{u}(r), \overline{u}(r)\right)$ of functions , $0 \le r \le 1$, which satisfies the following:

1) $\underline{u}(r)$ is bounded monotonic increasing left continuous function on (0,1] and right continuous at 0.

2) $\overline{u}(r)$ is bounded monotonic decreasing left continuous function on (0,1] and right continuous at 0.

3) $\underline{u}(r) \le \overline{u}(r), 0 \le r \le 1$.

If $\underline{u}(r) = \overline{u}(r), 0 \le r \le 1$ then α is a crisp number.

For arbitrary $u(r) = \left(\underline{u}(r), \overline{u}(r)\right)$, $v = \left(\underline{v}(r), \overline{v}(r)\right)$, and $k \in \mathbb{R}$ we

define addition, subtraction and multiplication operations:

1) $\underline{u\bar{+}v}(r) = \underline{u}(r)\bar{+}\underline{v}(r), \overline{u\bar{+}v}(r) = \overline{u}(r)\bar{+}\overline{v}(r)$

2) $\underline{uv}(r) = \min\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r)\}$,

$\overline{uv}(r) = \max\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r)\}$

3) $\underline{ku}(r) = k\left(\underline{u}(r)\right), \overline{ku}(r) = k\left(\overline{u}(r)\right)\ if\ k \geq 0$

4) $\overline{ku}(r) = k\left(\underline{u}(r)\right), \underline{ku}(r) = k\left(\overline{u}(r)\right)\ if\ k < 0$

Definition (1.12) [30]: For arbitrary fuzzy numbers $u, v \in E$ the quantity

$$D(u,v) = \sup_{0 \leq r \leq 1} \{\max|\underline{u}(r) - \underline{v}(r)|, \max|\overline{u}(r) - \overline{v}(r)|\}$$

defines the distance between u and v. It is shown that (E, D) is a complete metric space [48].

Definition (1.13) [12]: Let E be the set of all fuzzy numbers. A function $f : \mathbb{R} \to E$ is called a fuzzy-valued function.

Definition (1.14) [11]: The fuzzy function $f : [a,b] \to E$ is said to be continuous if for arbitrary fixed $t_0 \in [a,b]$ and $\epsilon > 0$ there exists $\delta > 0$ such that

$$|t - t_0| < \delta \Rightarrow D\left(f(t), f(t_0)\right) < \epsilon.$$

Definition (1.15) [8]: A fuzzy function $f: (a,b) \to E$ is differentiable at $t \in (a,b)$, if there exist $f(t+h) - f(t), f(t) - f(t-h)$, and an element $f'(t) \in E$ such that:

(i) for all $h > 0$ sufficiently small, the limits in metric D:

$$\lim_{h\to 0^+} D\left(\frac{f(t+h)-f(t)}{h}, f'(t)\right) = \lim_{h\to 0^+} D\left(\frac{f(t)-f(t-h)}{h}, f'(t)\right) = 0$$

(ii) for all $h < 0$ sufficiently small, the limits in metric D:

$$\lim_{h\to 0^-} D\left(\frac{f(t+h)-f(t)}{h}, f'(t)\right) = \lim_{h\to 0^-} D\left(\frac{f(t)-f(t-h)}{h}, f'(t)\right) = 0$$

Then $f'(t)$ is called the fuzzy derivative of $f\ at\ t$.

Definition (1.16) [40]: Suppose that $f : [a,b] \to E$ is a fuzzy function. For each partition $P = \{t_0, t_1, \dots, t_n\}$ of $[a,b]$ and for arbitrary $\mathcal{E}_i \in [t_{i-1}, t_i]$, $(1 \le i \le n)$, suppose

$$R_p = \sum_{i=1}^{n} f(\mathcal{E}_i)(t_{i-1} - t_i)$$

Then the definite integral of fuzzy function $f(t)$ over $[a,b]$ is defined by

$$\int_a^b f(t)dt = lim\, R_p \ , \max|t_i - t_{i-1}| \to 0$$

provided this limit exists.

If the fuzzy function $f(t)$ is continuous, then its definite integral exists and

$\overline{\int_a^b f(t,r)dt} = \int_a^b \overline{f}(t,r)dt\, , \underline{\int_a^b f(t,r)dt} = \int_a^b \underline{f}(t,r)dt.$

Note: we can define the integral of fuzzy function using different approaches such as Lebesgue integral concept and Riemann integral concept like in definition (1.15), both approaches give the same value if the fuzzy function $f(t)$ is continuous.

Definition (1.17) [12]: Let $u, v \in E$. If there exists $w \in E$ such that $u = v + w$, then w is called the H-difference of u, v and it is denoted by $u - v$. The notation H-difference is abbreviation for Hukuhara difference.

1.4 Fuzzy linear systems

We use system of linear equations to represent a lot of problems in various areas to be solvable. Now, if the parameters in the system are imprecise and not crisp, then we represent this uncertainty by fuzzy numbers, and the system of linear equations is called fuzzy linear system.

Definition (1.18) [61]: If $A = \left(a_{ij}\right)$, $1 \leq i, j \leq n$ is an $n \times n$ crisp coefficient matrix and $b_i \in E$, $1 \leq i, j \leq n$ are fuzzy numbers are given, then the following $n \times n$ system of equations

$$a_{11}\mathrm{x}_1 + a_{12}\mathrm{x}_2 + \cdots + a_{1\mathrm{n}}\mathrm{x}_\mathrm{n} = \mathrm{b}_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots \tag{1.2}$$

$$a_{1\mathrm{n}}\mathrm{x}_1 + a_{2\mathrm{n}}\mathrm{x}_2 + \cdots + a_{\mathrm{nn}}\mathrm{x}_\mathrm{n} = \mathrm{b}_\mathrm{n}$$

is called a fuzzy linear system.

Definition (1.19) [56]: The vector of fuzzy numbers $x = (x_1, x_2, \dots, x_n)^T$ given by $x_j = \left(\underline{x}_j(r), \overline{x}_j(r)\right), 1 \leq j \leq n, 0 \leq r \leq 1,$ is said to be the solution of the fuzzy linear system (1.2) if

$$\begin{cases} \underline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \underline{a_{ij}x_j} = \underline{b}_i \\ \overline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \overline{a_{ij}x_j} = \overline{b}_i \end{cases} \quad , i = 1,2,\dots,n \tag{1.3}$$

Replacing the fuzzy linear system (1.2) by an $2n \times 2n$ crisp linear system using (1.3) and the operation of fuzzy numbers, yields:

$$SX = b \text{ or } \begin{bmatrix} B & C \\ C & B \end{bmatrix} \begin{bmatrix} \underline{X} \\ -\overline{X} \end{bmatrix} = \begin{bmatrix} \underline{b} \\ -\overline{b} \end{bmatrix} \tag{1.4}$$

where $S = (s_{kl}), 1 \leq k , l \leq 2n$, and we determine s_{kl} as follows:

$a_{ij} \geq 0 \Rightarrow s_{ij} = a_{ij}, \; s_{i+m,j+n} = a_{ij},$

$a_{ij} < 0 \Rightarrow s_{i,j+n} = -a_{ij}, \; s_{i+m,j} = -a_{ij},$

and any other value of s_{kl} is zero and

$$X = \begin{bmatrix} \underline{X} \\ -\overline{X} \end{bmatrix} = \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ -\overline{x}_1 \\ \vdots \\ -\overline{x}_n \end{bmatrix}, b = \begin{bmatrix} \underline{b} \\ -\overline{b} \end{bmatrix} = \begin{bmatrix} \underline{b}_1 \\ \vdots \\ \underline{b}_n \\ -\overline{b}_1 \\ \vdots \\ -\overline{b}_n \end{bmatrix},$$

The matrix B contains the positive entries of A, the matrix C contains the absolute of the negative entries of A and $A = B - C$.

Then the solution of the crisp linear system (1.4) is the solution of the fuzzy linear system (1.2).

The crisp linear system (1.4) can be uniquely solved for X if and only if the coefficient matrix S is nonsingular. If matrix A is nonsingular then matrix S may be singular so the following theorem tells us when S is nonsingular.

Theorem (1.1) [20]: The matrix S is nonsingular if and only if $A = B - C$ and $B + C$ are both nonsingular.

We will consider the following example to show that even if S is nonsingular the solution of crisp linear system (1.4) does not define a fuzzy solution of fuzzy linear system (1.2).

Example (1.1): Consider the following fuzzy system

$$3x_1 - x_2 = b_1$$

$$2x_1 + x_2 = b_2$$

where $b_1(r) = \left(\underline{b}_1(r), \overline{b}_1(r)\right)$ and $b_2(r) = \left(\underline{b}_2(r), \overline{b}_2(r)\right)$. We briefly write $b_1 = \left(\underline{b}_1, \overline{b}_1\right)$ and $b_2 = \left(\underline{b}_2, \overline{b}_2\right)$.

$A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Then the extended matrix is $S = \begin{bmatrix} 3 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$, where $A = B - C$ and $B + C = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ are both nonsingular. Then by Theorem (1.1) S is nonsingular matrix. Therefore the linear crisp system (1.4) has a unique solution. The solution of system (1.4) for the matrix X:

$$SX = b$$

$$\begin{bmatrix} 3 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ -\overline{x}_1 \\ -\overline{x}_2 \end{bmatrix} = \begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \\ -\overline{b}_1 \\ -\overline{b}_2 \end{bmatrix}$$

$$\underline{x}_1 = \frac{3}{5}\left(\underline{b}_1 + \overline{b}_2\right) - \frac{2}{5}\left(\overline{b}_1 + \underline{b}_2\right)$$

$$\overline{x}_1 = \frac{2}{5}\left(\underline{b}_1 + \overline{b}_2\right) - \frac{3}{5}\left(\overline{b}_1 + \underline{b}_2\right)$$

$$\underline{x}_2 = \underline{b}_2 - \frac{6}{5}\left(\underline{b}_1 + \overline{b}_2\right) + \frac{4}{5}\left(\overline{b}_1 + \underline{b}_2\right)$$

$$\overline{x}_2 = -\overline{b}_2 - \frac{4}{5}\left(\underline{b}_1 + \overline{b}_2\right) + \frac{6}{5}\left(\overline{b}_1 + \underline{b}_2\right)$$

Note that $x_1 = \left(\underline{x}_1, \overline{x}_1\right)$ and $x_2 = \left(\underline{x}_2, \overline{x}_2\right)$ are not necessarily fuzzy numbers. From the condition $\underline{x}_1 < \overline{x}_1$ we get

$$\overline{b}_2 + \underline{b}_2 \leq -\left(\overline{b}_1 + \underline{b}_1\right) \qquad (1.5)$$

and from $\underline{x}_2 < \overline{x}_2$ we get

$$-\left(\overline{b}_1 - \underline{b}_1\right) \leq \overline{b}_2 - 9\underline{b}_2 \qquad (1.6)$$

Therefore $x_1 = \left(\underline{x}_1, \overline{x}_1\right)$ *and* $x_2 = \left(\underline{x}_2, \overline{x}_2\right)$ are fuzzy numbers if and only if the inequalities of (1.5) and (1.6) hold.

If the matrix S is nonsingular, then the solution vector X represent a solution fuzzy vector to the fuzzy linear system (1.2) if only if $\left(\underline{x}_j(r), \overline{x}_j(r)\right)$ is a fuzzy number for all j.

Definition (1.20) [11]: Let $X = \left\{\left(\underline{x}_j(r), \overline{x}_j(r)\right), 1 \leq j \leq n\right\}$ denotes the unique solution of (1.4), then the fuzzy number vector

$U = \left\{\left(\underline{u}_j(r), \overline{u}_j(r)\right), 1 \leq j \leq n\right\}$ defined by:

$$\underline{u}_j(r) = \min\{\underline{x}_j(r), \overline{x}_j(r), \underline{x}_j(1), \overline{x}_j(1)\}$$

$$\overline{u}_j(r) = \max\{\underline{x}_j(r), \overline{x}_j(r), \underline{x}_j(1), \overline{x}_j(1)\}$$

is called the fuzzy solution of (1.4).

If $\left(\underline{x}_j(r), \overline{x}_j(r)\right), 1 \leq j \leq n$ are all fuzzy numbers, then:

$\underline{u}_j(r) = \underline{x}_j(r), \overline{u}_j(r) = \overline{x}_j(r), 1 \leq j \leq n$ and U is called a strong fuzzy solution. Otherwise, U is called a weak fuzzy solution.

Now a necessary and sufficient condition for the existence of a strong fuzzy solution can be described by the following theorem:

Theorem (1.2) [20]: The system (1.4) has a strong fuzzy solution if and only if $(B+C)^{-1}(\underline{b} - \overline{b}) \leq 0$, where $S = \begin{bmatrix} B & C \\ C & B \end{bmatrix}$ is a nonsingular matrix.

By combining both theorems (1.1) and (1.2), then we get the following theorem:

Theorem (1.3) [20]: The fuzzy linear system (1.2) has a unique strong solution if and only if

1) The matrices $A = B - C$ and $B + C$ are both nonsingular.

2) $(B + C)^{-1}(\underline{b} - \overline{b}) \leq 0.$

1.5 Fuzzy integral equations

There are three major types of fuzzy integral equations:

1.5.1 The Fuzzy Fredholm integral equations of the second kind

A standard form of the Fredholm integral equation of the second kind is given by [24]:

$$g(t) = f(t) + \lambda \int_a^b k(s,t) g(s) ds \tag{1.7}$$

where λ is a positive parameter, $k(s,t)$ is a function called the kernel of the integral equation defined over the square $G: [a,b] \times [a,b]$ and $f(t)$ is a given function of $t \in [a,b]$.

Now, if $f(t)$ is a crisp function then (1.7) possess crisp solution and the solution is fuzzy if $f(t)$ is a fuzzy function.

With respect to definition (1.11) we introduce parametric form of a fuzzy Fredholm integral equation of the second kind. Let $\left(\underline{f}(t,r), \overline{f}(t,r)\right)$ and $\left(\underline{g}(t,r), \overline{g}(t,r)\right), 0 \leq r \leq 1 \; and \; t \in [a,b]$ are parametric forms of $f(t)$

and $g(t)$ respectively, then parametric form of fuzzy Fredholm integral equation of the second kind is as follows:

$$\underline{g}(t,r) = \underline{f}(t,r) + \lambda \int_a^b \underline{U}(s,r)ds$$

$$\overline{g}(t,r) = \overline{f}(t,r) + \lambda \int_a^b \overline{U}(s,r)ds \qquad (1.8)$$

where

$$\underline{U}(s,r) = \begin{cases} k(s,t)\underline{g}(s,r), & k(s,t) \geq 0 \\ k(s,t)\overline{g}(s,r), & k(s,t) < 0 \end{cases} \qquad (1.9)$$

and

$$\overline{U}(s,r) = \begin{cases} k(s,t)\overline{g}(s,r), & k(s,t) \geq 0 \\ k(s,t)\underline{g}(s,r), & k(s,t) < 0 \end{cases} \qquad (1.10)$$

for each $0 \leq r \leq 1$ and $a \leq s,t \leq b$. We can see that (1.8) is a crisp system of linear Fredholm integral equations for each $0 \leq r \leq 1$ and $a \leq t \leq b$.

Definition (1.21) [7]: The fuzzy Fredholm integral equations system of the second kind is of the form:

$$g_i(t) = f_i(t) + \sum_{j=1}^{m} \left(\lambda_{ij} \int_a^b k_{ij}(s,t) g_j(s) ds \right) \quad , i = 1, \ldots, m \qquad (1.11)$$

where s, t, λ are real constants and $s, t \in [a,b], \lambda_{ij} \neq 0$ for $i, j = 1, \ldots, m$.

In system (1.11), $g(t) = [g_1(t), \dots, g_m(t)]^T$ is unknown function, the fuzzy function $f_i(t)$ and kernel $k_{ij}(s,t)$ are known and we assume they are sufficiently differentiable with respect to all their arguments on the interval $[a,b]$.

Now, consider the parametric form of $f_i(t)$ and $g_i(t)$ to be $(\underline{f_i}(t,r), \overline{f}_i(t,r))$ and $(\underline{g_i}(t,r), \overline{g_i}(t,r))$,$0 \le r \le 1, t \in [a,b]$ respectively.

We write the parametric form of the given fuzzy Fredholm integral equations system as follows:

$$\begin{cases} \underline{g_i}(t,r) = \underline{f_i}(t,r) + \sum_{j=1}^{m} \left(\lambda_{ij} \int_a^b \underline{U}_{i,j}(s,r)ds \right) \\ \overline{g}_i(t,r) = \overline{f}_i(t,r) + \sum_{j=1}^{m} \left(\lambda_{ij} \int_a^b \overline{U}_{i,j}(s,r)ds \right) \end{cases} , i = 1, \dots, m \qquad (1.1\,2)$$

where

$$\underline{U}_{i,j}(s,r) = \begin{cases} k_{i,j}(s,t)\underline{g}_j(s,r), & k_{i,j}(s,t) \ge 0 \\ k_{i,j}(s,t)\overline{g}_j(s,r), & k_{i,j}(s,t) < 0 \end{cases} \qquad (1.13)$$

and

$$\overline{U}_{i,j}(s,r) = \begin{cases} k_{i,j}(s,t)\underline{g}_j(s,r), & k_{i,j}(s,t) \ge 0 \\ k_{i,j}(s,t)\overline{g}_j(s,r), & k_{i,j}(s,t) < 0 \end{cases} \qquad (1.14)$$

1.5.2 The Fuzzy Volterra integral equations of the second kind

The Volterra integral equations of the second kind is the same as Fredholm integral equations of the second kind (1.7) with slight difference; the upper limit of integration is variable.

Now, recall the Fredholm integral equation of the second kind (1.7). If the kernel function satisfies $k(s,t) = 0\,, s > t,$ then the Volterra integral equation has the general form:

$$g(t) = f(t) + \lambda \int_a^t k(s,t)g(s)ds \tag{1.15}$$

This equation has crisp solution if $f(t)$ is a crisp function. Equations (1.7) and (1.15) possess fuzzy solution if $f(t)$ is a fuzzy function.

Definition (1.22) [36]: The second fuzzy linear Volterra integral equations system is of the form:

$$g_i(t) = f_i(t) + \sum_{j=1}^{m} \left(\lambda_{ij} \int_a^t k_{ij}(s,t) g_j(s) ds \right) \quad , i = 1, \dots, m \tag{1.16}$$

where s, t, λ are real constants and $s, t \in [a, b], \lambda_{ij} \neq 0$ for $i, j = 1, \dots, m$. In system (1.17), $g(t) = [g_1(t), \dots, g_m(t)]^T$ is the solution to be determined. The fuzzy function $f_i(t)$ and kernel $k_{i,j}(s,t)$ are given and assumed to be sufficiently differentiable with respect to all their arguments on the interval $a \leq s, t \leq b$.

Now, let $\left(\underline{f_i}(t,r), \overline{f}_i(t,r)\right)$ and $\left(\underline{g_i}(t,r), \overline{g}_i(t,r)\right)$, $r \in [0,1], t \in [a,b]$ be parametric form of $f_i(t)$ and $g_i(t)$ respectively.

We assume that λ_{ij} is positive constant for $i, j = 1, \dots, m$, we write the parametric form of the given fuzzy Volterra integral equation system as follows:

$$\begin{cases} \underline{g}_i(t,r) = \underline{f}_i(t,r) + \sum_{j=1}^{m} \left(\lambda_{ij} \int_a^t \underline{U}_{i,j}(s,r)ds \right) \\ \overline{g}_i(t,r) = \overline{f}_i(t,r) + \sum_{j=1}^{m} \left(\lambda_{ij} \int_a^t \overline{U}_{i,j}(s,r)ds \right) \end{cases} \quad , i = 1, \dots, m \tag{1.17}$$

where

$$\underline{U}_{i,j}(s,r) = \begin{cases} k_{i,j}(s,t)\underline{g}_j(s,r), & k_{i,j}(s,t) \geq 0 \\ k_{i,j}(s,t)\overline{g}_j(s,r), & k_{i,j}(s,t) < 0 \end{cases} \tag{1.18}$$

and

$$\overline{U}_{i,j}(s,r) = \begin{cases} k_{i,j}(s,t)\underline{g}_j(s,r), & k_{i,j}(s,t) \geq 0 \\ k_{i,j}(s,t)\overline{g}_j(s,r), & k_{i,j}(s,t) < 0 \end{cases} \tag{1.19}$$

1.5.3 Fuzzy integro-differential equations

The fuzzy linear integro-differential equation [58]:

$$g'(t) = f(t) + \lambda \int_a^t k(s,t) g(s) ds \quad , g(t_0) = g_0 \tag{1.20}$$

where λ is positive constant, $k(s,t)$ is a function called kernel defined over the square $s \in [a,b] \times [a,b]$, and $f(t)$ is a given function of $t \in [a,b]$.

If g is a fuzzy function, $f(t)$ is a given fuzzy function of $t \in [a,b]$ and g' is the fuzzy derivative according to definition (1.17) of g. This equation may only possess fuzzy solution. If t is a variable and $k(s,t) = 0$ then (1.20) is fuzzy Volterra integro-differential equation of the second kind**.** If t is constant then (1.20) is fuzzy Fredholm integro-differential equation of the second kind.

Let $g(t) = \left(\underline{g}(t,r), \overline{g}(t,r)\right)$ is a fuzzy solution of (1.20), therefore using definitions (1.10), (1.13), and (1.15) we have the equivalent system:

$$\begin{cases} \underline{g}'(t) = \underline{f}(t) + \lambda \int_a^t \underline{U}(s,r)ds \ , \underline{g}(t_0) = \underline{g}_0 \\ \overline{g}'(t) = \overline{f}(t) + \lambda \int_a^t \overline{U}(s,r)ds \ , \overline{g}(t_0) = \overline{g}_0 \end{cases} \tag{1.21}$$

for each $r \in [0,1]$ and $s,t \in [a,b]$, where

$$\underline{U}(s,r) = \begin{cases} k(s,t)\underline{g}(s,r), & k(s,t) \geq 0 \\ k(s,t)\overline{g}(s,r), & k(s,t) < 0 \end{cases} \tag{1.22}$$

and

$$\overline{U}(s,r) = \begin{cases} k(s,t)\overline{g}(s,r), & k(s,t) \geq 0 \\ k(s,t)\underline{g}(s,r), & k(s,t) < 0 \end{cases} \tag{1.23}$$

Chapter Two

Existence and Uniqueness of Solutions of Fuzzy Fredholm Integral Equations

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Existence and Uniqueness of Solutions of Fuzzy Fredholm Integral Equations

Let $P(\mathbb{R})$ denotes the family of all nonempty compact convex subsets of $\mathbb{R}$ and define the addition and scalar multiplication in $P(\mathbb{R})$ as usual [51].

Definition (2.1) [42]: Let A and B are two nonempty bounded subsets of $\mathbb{R}$, then the distance in the Hausdorff metric between A and B is defined by: $d(A,B) = \max\{\sup_{b\in B} \inf_{a\in A} \|a-b\|, \sup_{a\in A} \inf_{b\in B} \|a-b\|\}$

where $\|.\|$ denotes the Euclidean norm in $\mathbb{R}$. Then $(P(\mathbb{R}), d)$ is a complete metric space [60].

Note: we use the notation I, where $I = [a,b] \subset \mathbb{R}$ and its closed bounded interval.

We use Zadeh's extension principle to extend the continuous function

$h: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ to $h: E^n \times E^n \to E^n$ as follow:

$$\bar{h}(u,v)(z) = \sup_{z=h(u,v)} \min\{u(x), v(x)\}$$

We know that:

$[\bar{h}(u,v)]^\alpha = h([u]^\alpha, [v]^\alpha)$ for all $u, v \in E, \alpha \in [0,1]$.

We can embed the real numbers in E using the rule $c \to \bar{c}(t)$ where

$$\bar{c}(t) = \begin{cases} 1 & for\ t = c \\ 0 & elsewhere \end{cases}$$

We define the addition and scalar multiplication operations as follows:

$[u+v]^{\alpha} = [u]^{\alpha} + [v]^{\alpha}$

$[\lambda u]^{\alpha} = \lambda [u]^{\alpha}$

where $u, v \in E, and\ \alpha \in [0,1]$.

Definition (2.2) [23]: Let $D : E^n \times E^n \to \mathbb{R}^+ \cup \{0\}$ and $u, v \in E$, then define D by:

$$D(u,v) = \sup_{0 \le \alpha \le 1} d([u]^{\alpha}, [v]^{\alpha})$$

where d is the Hausdorff metric defined in (2.1).

Definition (2.3) [34]: A mapping $g : I \to E^n$ is bounded, if there exists a constant $R > 0$ such that

$$D(g(t), \bar{0}) \le R, \forall t \in I.$$

For $u, v, w, z \in E$ and $\lambda \in \mathbb{R}$ we have the following properties on the metric D [41]:

1) $D(u + w, v + w) = D(u,v)$

2) $D(u \,\bar{*}\, v, \bar{0}) \le D(u, \bar{0}) D(v, \bar{0})$ for every $u, v, w \in E$ where the fuzzy multiplication $\bar{*}$ is based on the extension principle.

3) $D(\lambda u, \lambda v) = |\lambda| D(u,v)$

4) $D(u + v, w + z) \le D(u,w) + D(v,z)$.

Definition (2.4) [22]: A mapping $g : I \to E^n$ is strongly measurable if for all $\alpha \in [0,1]$ the set valued map $g_\alpha : I \to P(\mathbb{R})$ defined by:

$g_\alpha(t) = [g(t)]_\alpha$ is Lebesgue measurable when $P(\mathbb{R})$ has the topology induced by the Hausdorff metric d.

Definition (2.5) [48]: Let $g : I \to E^n$. If there is an integrable function h such that $\| x \| \leq h(t)$ for all $x \in g_0(t)$, then g is called integrable bounded.

Definition (2.6) [27]: Let $g : I \to E^n$. We define the integral levelwise by

$$\left[\int_a^b g(t)dt\right]^\alpha = \int_a^b g_\alpha(t)dt$$

$$= \left\{\int_a^b f(t)dt \,\middle|\, f: I \to \mathbb{R}^n \; is\ a\ measurable\ selection\ for\ g_\alpha\right\}$$

for all $\alpha \in [0,1]$.

Note: Consider a fuzzy mapping $g : I \to E^n$ that satisfy definitions (2.4) and (2.5) then g is integrable ($i.e., \int_a^b g(t)dt \in E$).

Theorem (2.1) [23]: A fuzzy mapping $g : I \to E^n$ is integrable if g is continuous.

Theorem (2.2) [27]: Let $F, G : I \to E^n$ be integrable and $\in \mathbb{R}$. Then

(1) $\int_a^b (F(t) + G(t)dt = \int_a^b F(t)dt + \int_a^b G(t)dt$

(2) $\int_a^b \lambda F(t)dt = \lambda \int_a^b F(t)dt$

(3) $D(F,G)$ is integrable

(4) $D(\int_a^b F(t)dt\,, \int_a^b G(t)dt) \leq \int_a^b D\big(F(t), G(t)\big)dt$

Theorem (2.3) [47]: Let the kernel $k : I \times I \to \mathbb{R}$ and the fuzzy function $f : I \to E^n$ are continuous functions and $\lambda \in \mathbb{R}$. Moreover, if

$M = \sup|k(t,s)|\,, t, s \in I$ and

$$|\lambda| < \frac{1}{M(b-a)} \tag{2.1}$$

then equation (1.7) has a unique continuous fuzzy solution on I.

Proof : let us define the sequence of continuous functions on I:

$$g_0(t) = f(t)$$

$$g_n(t) = f(t) + \lambda \int_a^b k(t,s) g_{n-1}(s)ds$$

then we have:

$$g_n(t) = f(t) + \lambda \int_a^b k(t,s)g(s)ds + \cdots + \lambda^n \int_a^b k_n(t,s)g(s)ds \tag{2.2}$$

where $k_1(t,s) = k(t,s)$ and

$$k_n(t,s) = \int_a^b k(t,u)k_{n-1}(u,s)du \quad , \quad n \geq 2 \tag{2.3}$$

in the virtue of equation (2.3) we have:

$$k_n(t,s) = \int_a^b \ldots \int_a^b k(t,t_1)k(t_1,t_2) \ldots k(t_{n-1},s)dt_1 \ldots dt_{n-1} \tag{2.4}$$

we get:

$$|k_n(t,s)| \leq M^n(b-a)^{n-1}, \quad n \geq 1 \tag{2.5}$$

equation (2.4) leads to the following formula:

$$k_n(t,s) = \int_a^b k_p(t,u)k_{n-p}(u,s)du, \qquad 1 \leq p \leq n-1 \tag{2.6}$$

Now equation (2.2) becomes:

$$g_n(t) = f(t) + \lambda \int_a^b \{\sum_{j=1}^n k_j(t,s)\lambda^{j-1}\} g(s)ds, \quad n \geq 1 \tag{2.7}$$

where the series

$$\sum_{j=1}^\infty k_j(t,s)\lambda^{j-1} \tag{2.8}$$

is uniformly convergent with respect to $(t,s) \in I \times I$ for $|\lambda| < \frac{1}{M(b-a)}$, equation (2.5) show that the series (2.8) is dominated by the general term $M^n(|\lambda|(b-a))^{n-1}$.

We set

$$P(t,s,\lambda) = \sum_{j=1}^\infty k_j(t,s)\lambda^{j-1} \tag{2.9}$$

where $P(t,s,\lambda)$ is a continuous function defined over $I \times I$ and $|\lambda| < \frac{1}{M(b-a)}$. Now, substituting equation (2.9) into equation (2.7) yields:

$$g(t) = f(t) + \int_a^b P(t,s,\lambda)g(s)ds \tag{2.10}$$

From (2.2) and its analogue for $n-1$, gives:

$$D\big(g_n(t), g_{n-1}(t)\big)$$

$$= D\Bigg(f(t) + \lambda \int_a^b k(t,s)g(s)ds + \cdots$$

$$+ \lambda^{n-1} \int_a^b k_{n-1}(t,s)g(s)ds + \lambda^n \int_a^b k_n(t,s)g(s)ds\,, f(t)$$

$$+ \lambda \int_a^b k(t,s)g(s)ds + \cdots + \lambda^{n-1} \int_a^b k_{n-1}(t,s)g(s)ds\Bigg)$$

$$\leq D\left(\lambda^n \int_a^b k_n(t,s)g(s)ds\,, \bar{0}\right) \leq |\lambda|^n D\left(\int_a^b k_n(t,s)g(s)ds\,, \bar{0}\right)$$

$$\leq |\lambda|^n \int_a^b D(k_n(t,s)g(s), \bar{0})ds \leq |\lambda|^n \int_a^b |\,k_n(t,s)|D(g(s), \bar{0})ds$$

$$\leq |\lambda|^n \sup_{a \leq t,s \leq b} |k_n(t,s)| \sup_{a \leq t \leq b} D(g(t), \bar{0}) \int_a^b 1.\,ds$$

Using theorem (2.2) and definition (2.3) yields:

$$\leq R|\lambda|^n M^n (b-a)^n = R\big(|\lambda| M(b-a)\big)^n, \qquad n \geq 1 \tag{2.11}$$

with $\big(|\lambda| M(b-a)\big)^n < 1$.

Then the sequence $\{g_n(t)\}$ is uniformly convergent on $[a,b]$, where

$g(t) = \lim_{n\to\infty} g_n(t)$, so $g(t)$ is a solution of equation (1.7).

To prove the uniqueness we assume that $y(t)$ is a solution of equation (1.7), that is, $y(t) = f(t) + \lambda \int_a^b k(s,t)y(s)ds$

Using the recurrence formula for $g_n(t)$, we obtain:

$$D\big(y(t), g_n(t)\big) = D\left(f(t) + \lambda\int_a^b k(t,s)y(s)ds\,, f(t) + \lambda\int_a^b k(t,s)g_{n-1}(s)ds\right)$$

$$\leq \mathrm{D}\left(\lambda\int_a^b k(t,s)y(s)ds, \lambda\int_a^b k(t,s)g_{n-1}(s)ds\right)$$

$$\leq |\lambda|\mathrm{D}\left(\int_a^b k(t,s)y(s)ds, \int_a^b k(t,s)g_{n-1}(s)ds\right)$$

$$\leq |\lambda|\int_a^b D\big(k(t,s)y(s), k(t,s)g_{n-1}(s)\big)ds$$

$$\leq |\lambda|\int_a^b |k(t,s)|D\big(y(s), g_{n-1}(s)\big)ds$$

$$\leq |\lambda| \sup_{a\leq t,s\leq b} |k(t,s)| \sup_{a\leq t\leq b} D\big(y(t), g_{n-1}(t)\big)\int_a^b 1.\,ds$$

Hence we get

$$\sup_{a\leq t\leq b} D\big(y(t), g_n(t)\big) \leq |\lambda|M(b-a) \sup_{a\leq t\leq b} D\big(y(t), g_{n-1}(t)\big)\,, n \geq 1 \quad (2.12)$$

If we denote $\alpha_n = \sup_{a\leq t\leq b} D\big(y(t), g_n(t)\big)$, then (2.12) becomes

$$\alpha_n = |\lambda|M(b-a)\alpha_{n-1}$$

and finally

$$\alpha_n = (|\lambda|M(b-a))^n\alpha_0\,, \qquad n \geq 1$$

Condition (2.1) shows that $\alpha_n \to 0$ as $n \to \infty$, i.e.

$$y(t) = \lim_{n \to \infty} g_n(t) = g(t) \qquad , t \in [a, b].$$

Chapter Three

Analytical Methods for Solving Fuzzy Fredholm Integral Equation of the Second Kind

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3.1 Fuzzy Laplace Transform Method

In this section, we introduce some basic definitions for the fuzzy Laplace transform and fuzzy convolution. Then we apply fuzzy Laplace transforms to solve fuzzy convolution Fredholm integral equation of the second kind.

Theorem (3.1) [62]: Let $f(t)$ be fuzzy-valued function on $[0,\infty)$ represented by $\left(\underline{f}(t,r),\overline{f}(t,r)\right)$. For any fixed $r\in[0,1]$, assume $\underline{f}(t,r)$ and $\overline{f}(t,r)$ are Riemann-Integrable on $[a,b]$ for every $b\geq a$, and assume there are two positive $\underline{M}(r)$ and $\overline{M}(r)$ such that $\int_a^b |\underline{f}(t,r)|\,dt\leq\underline{M}(r)$ and $\int_a^b |\overline{f}(t,r)|\,dt\leq\overline{M}(r)$ for every $b\geq a$. Then $f(t)$ is improper fuzzy Riemann-integrable on $[0,\infty)$ and it is a fuzzy number. Furthermore, we have:

$$\int_a^\infty f(t)dt=\left(\int_a^\infty \underline{f}\,(t,r)dt,\int_a^\infty \overline{f}\,(t,r)dt\right)$$

Proof: see [62].

Proposition (3.1) [61]: If $f(t)$ and $g(t)$ are fuzzy-valued functions and fuzzy Riemann-integrable on $[a,\infty)$, then $f(t)+g(t)$ is also Riemann-integrable on $[a,\infty), i.e.$

$$\int_a^\infty \big(f(t)+g(t)\big)\,dt = \int_a^\infty f(t)\,dt + \int_a^\infty g(t)\,dt$$

Definition (3.1) [50]: Let $f(t)$ be fuzzy-valued function and s is real parameter. Then we define the fuzzy Laplace transforms as follows:

$$F(s) = L\big(f(t)\big) = \int_0^\infty e^{-st} f(t)\,dt = \lim_{\tau\to\infty}\int_0^\tau e^{-st} f(t)\,dt \qquad (3.1)$$

using Theorem (3.1), we have

$$F(s) = \left(\lim_{\tau\to\infty}\int_0^\tau e^{-st}\underline{f}(t)\,dt\,,\lim_{\tau\to\infty}\int_0^\tau e^{-st}\overline{f}(t)\,dt\right)$$

whenever the limits exists, and we denote the fuzzy Laplace transform by L, that generate a new fuzzy-valued function, $F(s) = L\big(f(t)\big)$.

Now, consider the fuzzy-valued function f then $l\left(\underline{f}(t,r)\right)$ and $l\left(\overline{f}(t,r)\right)$ the lower and upper fuzzy Laplace transform respectively, we use the definition of Laplace transform $, i.e.$

$$l\left(\underline{f}(t,r)\right) = \int_0^\infty e^{-st}\underline{f}(t,r)\,dt = \lim_{\tau\to\infty}\int_0^\tau e^{-st}\underline{f}(t,r)\,dt$$

$$l\left(\overline{f}(t,r)\right) = \int_0^\infty e^{-st}\overline{f}(t,r)\,dt = \lim_{\tau\to\infty}\int_0^\tau e^{-st}\overline{f}(t,r)\,dt$$

then we get

$$F(s,r) = L\big(f(t,r)\big) = \left[l\left(\underline{f}(t,r)\right), l\left(\overline{f}(t,r)\right)\right]$$

Theorem (3.2) [2]: Let f and g are continuous fuzzy-valued functions and c_1, c_2 are constants. Then:

$$L\big[\big(c_1.f(t)\big) + \big(c_2.g(t)\big)\big] = (c_1.L[f(t)]) + (c_2.L[g(t)])$$

Proof: $L\big[\big(c_1.f(t)\big) + \big(c_2.g(t)\big)\big] = \int_0^\infty \Big(\big(c_1.f(t)\big) + \big(c_2.g(t)\big)\Big).e^{-st}\,dt$

$$= \int_0^\infty \big(c_1.f(t)\big).e^{-st}dt + \int_0^\infty \big(c_2.g(t)\big).e^{-st}dt$$

$$= c_1.\int_0^\infty f(t).e^{-st}dt + c_2.\int_0^\infty g(t).e^{-st}dt = (c_1.L[f(t)]) + (c_2.L[g(t)])$$

thus,

$$L\big[\big(c_1.f(t)\big) + \big(c_2.g(t)\big)\big] = (c_1.L[f(t)]) + (c_2.L[g(t)])$$

Lemma (3.1) [10]: Let f be continuous fuzzy valued function on $[0,\infty)$ and $\lambda \in \mathbb{R}$. Then:

$$L[\lambda.f(t)] = \lambda.L[f(t)]$$

Theorem (3.3) [10]: (First translation theorem) Let f be continuous fuzzy-valued function and $L[f(t)] = F(s)$, then:

$$L[e^{at}.f(t)] = F(s-a)$$

where e^{at} is real valued function and $s - a > 0$.

Proof: $L[e^{at}.f(t)] = \int_0^\infty e^{at}.e^{-st}.f(t)dt = \int_0^\infty e^{at-st}.f(t)dt$

$$= \int_{0}^{\infty} e^{-(s-a)t}.f(t)dt = F(s-a)$$

hence,

$$L[e^{at}.f(t)] = F(s-a)$$

Definition (3.2) [15]: Let f be fuzzy-valued function. If $\lim_{t\to t_0^-} f(t)$, $\lim_{t\to t_0^+} f(t)$ exist and $f(t_0^-) \neq f(t_0^+)$ then f has a jump discontinuity at t_0.

Definition (3.3): The fuzzy-valued function f is piecewise continuous on $[0,\infty)$ if:

1) $\lim_{t\to 0^+} f(t) = f(0^+)$ exists.

2) f is continuous on every finite interval $(0,b)$ expect possibly at a finite number of points $\tau_1, \tau_2, \cdots, \tau_n$ in $(0,b)$ at which f has jump discontinuity.

Definition (3.4): Let f be fuzzy-valued function and M is positive constant, then f is bounded if $|f(t)| \leq M_i, \tau_i \leq t \leq \tau_{i+1}, i = 1, \cdots, n-1$.

Definition (3.5): Let f be fuzzy-valued function. If there exist constants $M > 0$ and p such that for some $t_0 \geq 0$, if $|f(t)| \leq Me^{pt}.\overline{1}, t \geq t_0$, then f has exponential order p.

Theorem (3.4) [15]: If the fuzzy-valued function f is piecewise continuous on $[0,\infty)$ and has exponential order p, then

$F(s) = L[f(t)] \to 0, as \quad s \to \infty.$

Proof: see [15].

Theorem (3.5) [15]: Let f be fuzzy-valued function. If f is bounded and piecewise continuous on $[0,\infty)$ of exponential order p, then the fuzzy Laplace transform $F(s) = L[f(t)]$ exists for $s - a > 0$ and converges absolutely.

Proof: see [15].

The inverse of fuzzy Laplace transform $F(s) = L[f(t)]$ maps the fuzzy Laplace transform of a fuzzy-valued function f back to original fuzzy-valued function f, and we denote the inverse of fuzzy Laplace transform by

$L^{-1}\big(F(s)\big) = f(t), t \geq 0.$

The most important basic properties of the fuzzy Laplace transform and its inverse is linearity.

3.1.1 Fuzzy Convolution

It is important to introduce fuzzy convolution in order to solve fuzzy convolution Fredholm integral equation directly.

Definition (3.6): If $f(t)$ and $g(t)$ are two piecewise continuous fuzzy-valued functions, then their fuzzy convolution is defined by:

$$(f * g)(t) = \int_0^t f(t-s)\, g(s) ds.$$

Some properties of the fuzzy convolution:

1) $(f * g) = (g * f)$ (commutative)

2) $c.(f * g) = (c.f) * g = f * (c.g)$, c is constant

3) $f * (g * h) = (f * g) * h$ (associative)

We will introduce Convolution Theorem which is very important, it connect fuzzy convolution with fuzzy Laplace transform as follows:

Theorem (3.6) [50]: (Convolution Theorem) Let $f(t)$ and $g(t)$ be piecewise continuous fuzzy-valued functions on $[0,\infty)$ of exponential order p with fuzzy Laplace transform $F(s)$ and $G(s)$ respectively, $i.e.$

$L[f(t)] = F(s)$ and $L[g(t)] = G(s)$. Then

$$L[(f * g)(t)] = L[f(t)].L[g(t)] = F(s).G(s), s - p > 0$$

Proof: see [50].

Definition (3.7): The fuzzy convolution Fredholm integral equation of the second kind is defined as

$$g(t) = f(t) + \lambda \int_a^b k(s-t)\, g(s)ds \ , t \in [a,b], b < \infty \qquad (3.2)$$

where $f(t)$ is a continuous fuzzy-valued function and $k(s-t)$ is an arbitrary real-valued function called real-valued convolution kernel function.

Now, take fuzzy Laplace transform on both sides of equation (3.2), we get

$$L[g(t)] = L\left[f(t) + \lambda \int_a^b k(s-t)\, g(s) ds\right] \ ,t,s \in [a,b], b < \infty$$

using Theorem (3.2), we have

$$L[g(t)] = L[f(t)] + L\left[\lambda \int_a^b k(s-t)\, g(s) ds\right] \ ,t,s \in [a,b], b < \infty$$

Then, we use the definition of fuzzy Laplace transform:

$$l\left[\underline{g}(t,r)\right] = l\left[\underline{f}(t,r)\right] + l\left[\lambda \int_a^b k(s-t)\underline{g}(s,r)\, ds\right]$$

$$l[\overline{g}(t,r)] = l\left[\overline{f}(t,r)\right] + l\left[\lambda \int_a^b k(s-t)\overline{g}(s,r)\, ds\right]$$

we apply fuzzy convolution, we get

$$l\left[\underline{g}(t,r)\right] = l\left[\underline{f}(t,r)\right] + \lambda l[k(s,t)] l\left[\underline{g}(t,r)\right] \ ,r \in [0,1]$$

$$l[\overline{g}(t,r)] = l\left[\overline{f}(t,r)\right] + \lambda l[k(s,t)] l[\overline{g}(t,r)] \ ,r \in [0,1]$$

we consider the following two cases for the changing sign of $k(s,t)$:

Case (1): If $k(s,t) > 0$, then we get

$$l\left[\underline{g}(t,r)\right] = l\left[\underline{f}(t,r)\right] + \lambda l[k(s,t)] l\left[\underline{g}(t,r)\right]$$

$$l[\overline{g}(t,r)] = l\left[\overline{f}(t,r)\right] + \lambda l[k(s,t)] l[\overline{g}(t,r)]$$

Case (2): If $k(s,t) < 0$, then we get

$$l\left[\underline{g}(t,r)\right] = l\left[\underline{f}(t,r)\right] + \lambda l[k(s,t)] l[\overline{g}(t,r)]$$

$$l[\overline{g}(t,r)] = l\left[\overline{f}(t,r)\right] + \lambda l[k(s,t)] l\left[\underline{g}(t,r)\right]$$

For case (1) we find the explicit formulas that is

$$l\left[\underline{g}(t,r)\right] = \frac{l\left[\underline{f}(t,r)\right]}{1 - \lambda l[k(s,t)]}$$

$$l[\overline{g}(t,r)] = \frac{l\left[\overline{f}(t,r)\right]}{1 - \lambda l[k(s,t)]}$$

and the explicit formulas for case (2) as follow:

$$l\left[\underline{g}(t,r)\right] = \frac{l\left[\underline{f}(t,r)\right] + \lambda l[k(s,t)] l\left[\overline{f}(t,r)\right]}{1 - \lambda^2 l[k(s,t)] l[k(s,t)]}$$

$$l[\overline{g}(t,r)] = \frac{l\left[\overline{f}(t,r)\right] + \lambda l[k(s,t)] l\left[\underline{f}(t,r)\right]}{1 - \lambda^2 l[k(s,t)] l[k(s,t)]}$$

Finally, we take the inverse of the fuzzy Laplace transform we get the following for the first case:

$$\underline{g}(t,r) = l^{-1}\left(\frac{l\left[\underline{f}(t,r)\right]}{1 - \lambda l[k(s,t)]}\right)$$

$$\overline{g}(t,r) = l^{-1}\left(\frac{l\left[\overline{f}(t,r)\right]}{1 - \lambda l[k(s,t)]}\right)$$

and take the inverse of fuzzy Laplace transform for the second case, we get the following:

$$\underline{g}(t,r) = l^{-1}\left(\frac{l\left[\underline{f}(t,r)\right] + \lambda l[k(s,t)]l[\overline{f}(t,r)]}{1-\lambda^2 l[k(s,t)]l[k(s,t)]}\right)$$

$$\overline{g}(t,r) = l^{-1}\left(\frac{l[\overline{f}(t,r)] + \lambda l[k(s,t)]l\left[\underline{f}(t,r)\right]}{1-\lambda^2 l[k(s,t)]l[k(s,t)]}\right)$$

Example (3.1): Consider the following fuzzy convolution Fredholm integral equations

$$\underline{g}(t,r) = \frac{1}{2}(r+1).t + \int_0^2 \frac{1}{4}(t-s)\underline{g}(s,r)ds$$

$$\overline{g}(t,r) = \frac{1}{2}(3-r).t + \int_0^2 \frac{1}{4}(t-s)\overline{g}(s,r)ds$$

we apply the fuzzy Laplace transform to both sides of the equations, we get

$$l\left\{\underline{g}(t,r)\right\} = \frac{1}{2}l\{(r+1).t\} + \frac{1}{4}l\{t\}.l\left\{\underline{g}(t,r)\right\}, r \in [0,1]$$

$$l\{\overline{g}(t,r)\} = \frac{1}{2}l\{(3-r).t\} + \frac{1}{4}l\{t\}.l\{\overline{g}(t,r)\}, r \in [0,1]$$

then,

$$l\left\{\underline{g}(t,r)\right\} = \frac{2(r+1)}{4s^2-1}$$

$$l\{\overline{g}(t,r)\} = \frac{2(3-r)}{4s^2-1}$$

finally, by applying the inverse of fuzzy Laplace transform on both sides, we have

$$\underline{g}(t,r) = (r+1).\sinh\left(\frac{t}{2}\right)$$

$$\overline{g}(t,r) = (3-r).\sinh\left(\frac{t}{2}\right)$$

3.2 Homotopy Analysis Method (HAM)

The homotopy analysis method is considered as an analytical approach in order to obtain solutions in series form and use it to solve various types of integral equations. Then will ensure the convergence of the solution series using an auxiliary parameter. Moreover, homotopy analysis method provides a kind of freedom for choosing initial approximations and an auxiliary linear operator which helps us to simplify any problem.

Definition (3.8) [52]: Let ϕ be a function of the homotopy parameter p from homotopy theory. Then the nth-order homotopy-derivative of ϕ is defined by

$$D_n(\phi) = \frac{1}{n!}\frac{\partial^n \phi}{\partial p^n}\bigg|_{p=0} \tag{3.3}$$

where $n \geq 0$ is an integer.

Lemma (3.2) [53]: suppose $\phi = \sum_{n=0}^{\infty} u_n(t,r)\, p^n$ denote a homotopy series, where $p \in [0,1]$ is the embedding homotopy parameter in the theory of topology, u_n is an unknown function, where t and r denote a spatial and temporal independent variables respectively. Let $\mathcal{L}$ denote an auxiliary linear operator, and u_0 an initial guess solution. It holds that

$$D_n\{(1-p)\mathcal{L}[\phi - u_0]\} = \mathcal{L}[u_n(t,r) - \chi_n u_{n-1}(t,r)] \tag{3.4}$$

where D_{n-1} is defined by equation (3.3) and χ_n defined by

$$\chi_n = \begin{cases} 0, & n \leq 1 \\ 1, & n > 1 \end{cases} \tag{3.5}$$

Theorem (3.7) [52]: Let $\mathcal{L}$ be a linear operator independent of the homotopy-parameter p. For homotopy series

$$\phi = \sum_{k=0}^{\infty} u_k p^k$$

it holds

$$D_n\{\mathcal{L}[\phi]\} = \mathcal{L}[D_n(\phi)]$$

Proof: see [52].

Theorem (3.8) [53]: Let $\phi = \sum_{n=0}^{\infty} u_n(t,r)p^n$, where $p \in [0,1]$ is the homotopy parameter. Let $\mathcal{L}$ denote an auxiliary linear operator, $\mathcal{N}$ is a nonlinear operator, $u_0(t,r)$ an initial guess solution, h the convergence-control parameter, and $H(t,r)$ an auxiliary function, both h and $H(t,r)$ are independent of p. Then we define the $zero$-order deformation equation as follows:

$$(1-p)\mathcal{L}[\phi - u_0] = phH(t,r)\mathcal{N}[\phi],$$

the corresponding nth-order deformation equation ($n \geq 1$)

$$\mathcal{L}[u_n(t,r) - \chi_n u_{n-1}(t,r)] = hH(t,r)D_{n-1}(\mathcal{N}[\phi]) \tag{3.6}$$

where D_{n-1} and χ_n are defined by equations (3.3) and (3.5) respectively.

Finding the solution as a series form we need to investigate its convergence in any region, so we have the following theorem:

Theorem (3.9) [5]: (Convergence Theorem) If the series $u_0(t,r) + \sum_{n=1}^{\infty} u_n(t,r)$ converges to function $u(t,r)$, then $u(t,r)$ must be the exact solution, where $u_n(t,r)$ is governed by the nth-order deformation equation (3.6) under the definition (3.3) and equation (3.5).

Proof: see [53].

To ensure convergence of the series we have to concentrate on choosing $u_0(t,r)$ the initial guess, the linear operator $\mathcal{L}$, the embedding parameter p, the auxiliary parameter h, and finally the auxiliary function $H(t,r)$.

In this part, we rewrite the fuzzy Fredholm integral equations of the second kind, and then solve them by homotopy analysis method. Also, we get the solution in series form.

Now, we partition the interval $[a,b]$ into two parts according to the sign of the kernel $k(s,t)$, $i.e.$ $k(s,t) > 0$ on $[a,c]$ and $k(s,t) < 0$ on $[c,b]$. Therefore, we rewrite equation (1.8) as follow:

$$\underline{g}(t,r) = \underline{f}(t,r) + \lambda \int_a^c k(s,t)\underline{g}(s,r)ds + \lambda \int_c^b k(s,t)\overline{g}(s,r)ds$$

$$\overline{g}(t,r) = \overline{f}(t,r) + \lambda \int_a^c k(s,t)\overline{g}(s,r)ds + \lambda \int_c^b k(s,t)\underline{g}(s,r)ds \qquad (3.7)$$

From system (3.7) we define the nonlinear operator $\mathcal{N}(t,p,r)$ as follows [38]:

$$\underline{\mathcal{N}}(t,p,r) = \underline{U}(t,p,r) - \underline{f}(t,r) - \lambda \int_a^c k(s,t)\underline{U}(s,p,r)ds$$
$$- \lambda \int_c^b k(s,t)\overline{U}(s,p,r)ds$$

$$\overline{\mathcal{N}}(t,p,r) = \overline{U}(t,p,r) - \overline{f}(t,r) - \lambda \int_a^c k(s,t)\overline{U}(s,p,r)ds$$
$$- \lambda \int_c^b k(s,t)\underline{U}(s,p,r)ds \tag{3.8}$$

We choose the auxiliary linear operator $\mathcal{L}$ with the following assumption:

$$\mathcal{L}[U(t,p,r)] = U(t,p,r) \tag{3.9}$$

Applying homotopy analysis method to solve system (3.7), we consider the $zero$-order deformation equation

$$(1-p)\mathcal{L}\left[\underline{U}(t,p,r) - \underline{u}_0(t,r)\right] = ph\underline{H}(t,r)\underline{\mathcal{N}}(t,r)$$

$$(1-p)\mathcal{L}\left[\overline{U}(t,p,r) - \overline{u}_0(t,r)\right] = ph\overline{H}(t,r)\overline{\mathcal{N}}(t,r) \tag{3.10}$$

where $p \in [0,1]$ is an embedding parameter called the homotopy parameter, $\mathcal{L}$ is an auxiliary linear parameter, $\underline{u}_0(t,r)$ and $\overline{u}_0(t,r)$are the initial guess of $\underline{g}(t,r)$ and $\overline{g}(t,r)$ respectively, $\underline{H}(t,r) \neq 0$ and $\overline{H}(t,r) \neq 0$ are auxiliary functions, $\underline{U}(t,p,r)$ and $\overline{U}(t,p,r)$ are the unknown functions on independent variable p, and $h \neq 0$ denote convergence-controller parameter.

Now, applying equation (3.9) on equation (3.10), yields

$$(1-p)\left[\underline{U}(t,p,r) - \underline{u}_0(t,r)\right] = ph\underline{H}(t,r)\underline{\mathcal{N}}(t,r)$$

$$(1-p)\left[\overline{U}(t,p,r) - \overline{u}_0(t,r)\right] = ph\overline{H}(t,r)\overline{\mathcal{N}}(t,r)$$

then substitute equation (3.8) with the assumption that $H(t,r)=1$, we get [35]

$$(1-p)\left[\underline{U}(t,p,r)-\underline{u}_0(t,r)\right] = ph\left[\underline{U}(t,p,r)-\underline{f}(t,r)-\lambda\int_a^c k(s,t)\underline{U}(s,p,r)ds - \lambda\int_c^b k(s,t)\overline{U}(s,p,r)ds\right]$$

$$(1-p)\left[\overline{U}(t,p,r)-\overline{u}_0(t,r)\right] = ph\left[\overline{U}(t,p,r)-\overline{f}(t,r)-\lambda\int_a^c k(s,t)\overline{U}(s,p,r)ds - \lambda\int_c^b k(s,t)\underline{U}(s,p,r)ds\right] \quad (3.11)$$

when $p=0$, the *zero*-order deformation (3.11) becomes

$$\underline{U}(t,0,r)=\underline{u}_0(t,r)$$

$$\overline{U}(t,0,r)=\overline{u}_0(t,r) \quad (3.12)$$

when $p=1$, the *zero*-order deformation (3.11) becomes

$$\underline{U}(t,1,r)=\underline{f}(t,r)+\lambda\int_a^c k(s,t)\underline{U}(s,1,r)ds+\lambda\int_c^b k(s,t)\overline{U}(s,1,r)ds$$

$$\overline{U}(t,1,r)=\overline{f}(t,r)+\lambda\int_a^c k(s,t)\overline{U}(s,1,r)ds+\lambda\int_c^b k(s,t)\underline{U}(s,1,r)ds$$

$$(3.13)$$

Notice that equation (3.13) is exactly the same as equation (3.7).

Now as the value of p increases from 0 to 1 the analytical solution $\left(\underline{U}(t,p,r),\overline{U}(t,p,r)\right)$ changes from the initial approximation guess $\left(\underline{u}_0(t,r),\overline{u}_0(t,r)\right)$ to the exact solution $\left(\underline{g}(t,r),\overline{g}(t,r)\right)$.

We expand the functions $\underline{U}(t,p,r)$ and $\overline{U}(t,p,r)$ in a Taylor series with respect to the embedding parameter p.This expansion can be written as follows [52]:

$$\underline{U}(t,p,r) = \underline{u}_0(t,r) + \sum_{n=1}^{\infty} \underline{u}_n(t,r)\, p^n$$

$$\overline{U}(t,p,r) = \overline{u}_0(t,r) + \sum_{n=1}^{\infty} \overline{u}_n(t,r)\, p^n \tag{3.14}$$

where

$$\underline{u}_n(t,r) = \frac{1}{n!}\frac{\partial^n \underline{U}(t,p,r)}{\partial p^n}\bigg|_{p=0}$$

$$\overline{u}_n(t,r) = \frac{1}{n!}\frac{\partial^n \overline{U}(t,p,r)}{\partial p^n}\bigg|_{p=0} \tag{3.15}$$

We differentiate the $zero$-order deformation equation (3.11) n-times with respect to p, we get:

$$\begin{aligned}\frac{\partial^n \underline{U}(t,p,r)}{\partial p^n} - \frac{\partial^{n-1} \underline{U}(t,p,r)}{\partial p^{n-1}} &= h\left[\frac{\partial^{n-1} \underline{U}(t,p,r)}{\partial p^{n-1}} - \underline{f}(t,r) - \lambda \int_a^c k(s,t)\,\frac{\partial^{n-1} \underline{U}(s,p,r)}{\partial p^{n-1}}\,ds\right.\\ &\left. - \lambda \int_c^b k(s,t)\,\frac{\partial^{n-1} \underline{U}(s,p,r)}{\partial p^{n-1}}\,ds\right]\end{aligned}$$

$$\frac{\partial^n \overline{U}(t,p,r)}{\partial p^n} - \frac{\partial^{n-1}\overline{U}(t,p,r)}{\partial p^{n-1}}$$
$$= h\left[\frac{\partial^{n-1}\overline{U}(t,p,r)}{\partial p^{n-1}} - \overline{f}(t,r) - \lambda\int_a^c k(s,t)\frac{\partial^{n-1}\overline{U}(s,p,r)}{\partial p^{n-1}}ds\right.$$
$$\left. - \lambda\int_c^b k(s,t)\frac{\partial^{n-1}\overline{U}(s,p,r)}{\partial p^{n-1}}ds\right] \tag{3.16}$$

Dividing (3.16) by $n!$, then set $p = 0$, we get the nth-order deformation equation [38]:

$$\underline{u}_n(t,r) = \alpha_n \underline{u}_{n-1}(t,r)$$
$$+ h\left[\underline{u}_{n-1}(t,r) - \beta_n \underline{f}(t,r) - \lambda\int_a^c k(s,t)\underline{u}_{n-1}(s,p,r)ds\right.$$
$$\left. - \lambda\int_c^b k(s,t)\overline{u}_{n-1}(s,p,r)ds\right]$$

$$\overline{u}_n(t,r) = \alpha_n \overline{u}_{n-1}(t,r)$$
$$+ h\left[\overline{u}_{n-1}(t,r) - \beta_n \overline{f}(t,r) - \lambda\int_a^c k(s,t)\overline{u}_{n-1}(s,p,r)ds\right.$$
$$\left. - \lambda\int_c^b k(s,t)\overline{u}_{n-1}(s,p,r)ds\right] \tag{3.17}$$

where $n \geq 1$ and

$$\alpha_n = \begin{cases} 0, & n = 1 \\ 1, & n \neq 1 \end{cases}, \qquad \beta_n = \begin{cases} 0, & m \neq 1 \\ 1, & m = 1 \end{cases}$$

If we let $\underline{u}_0(t,r) = \overline{u}_0(t,r) = \overline{0}$, then for $n \geq 2$ we have:

$$\underline{u}_n(t,r) = (1+h)\underline{u}_{n-1}(t,r)$$
$$- h\lambda\left[\int_a^c k(s,t)\underline{u}_{n-1}(s,p,r)ds + \int_c^b k(s,t)\overline{u}_{n-1}(s,p,r)ds\right]$$

$$\overline{u}_n(t,r) = (1+h)\overline{u}_{n-1}(t,r) - h\lambda\left[\int_a^c k(s,t)\overline{u}_{n-1}(s,p,r)ds + \int_c^b k(s,t)\overline{u}_{n-1}(s,p,r)ds\right] \tag{3.18}$$

The solution of equation (3.7) in series form can be obtained as follows:

$$\underline{g}(t,r) = \lim_{p\to 1}\underline{U}(t,p,r) = \sum_{n=1}^{\infty}\underline{u}_n(t,r)$$

$$\overline{g}(t,r) = \lim_{p\to 1}\overline{U}(t,p,r) = \sum_{n=1}^{\infty}\overline{u}_n(t,r) \tag{3.19}$$

Now, we denote the mth-order approximation to solution $\underline{g}(t,r)$ with

$$\underline{g}_m(t,r) = \sum_{n=1}^{m}\underline{u}_n(t,r)$$

and $\overline{g}(t,r)$ with

$$\overline{g}_m(t,r) = \sum_{n=1}^{m}\overline{u}_n(t,r) \tag{3.20}$$

Example (3.2): Consider the fuzzy Fredholm integral equation (1.8) with

$$\underline{f}(t,r) = \frac{1}{2}(r+1).t$$

$$\overline{f}(t,r) = \frac{1}{2}(3-r).t$$

and

$$k(s,t) = \frac{1}{4}t \quad ,0 \le s,t \le 2$$

on the interval $[0,2], \lambda = 1$. The first terms of homotopy series are:

$$\underline{u}_0(t,r) = \bar{0}$$

$$\underline{u}_1(t,r) = -h\underline{f}(t,r) = -\frac{1}{2}ht(r+1)$$

$$\underline{u}_2(t,r) = (1+h)\underline{u}_1(t,r) - h\int_0^2 \frac{1}{4}t\underline{u}_1(s,r)\,ds = -\frac{1}{4}ht(r+1)(2+h)$$

$$\underline{u}_3(t,r) = (1+h)\underline{u}_2(t,r) - h\int_0^2 \frac{1}{4}t\underline{u}_2(s,r)\,ds = -\frac{1}{8}ht(r+1)(2+h)^2$$

$$\underline{u}_4(t,r) = (1+h)\underline{u}_3(t,r) - h\int_0^2 \frac{1}{4}t\underline{u}_3(s,r)\,ds = -\frac{1}{16}ht(r+1)(2+h)^3$$

$$\underline{u}_5(t,r) = (1+h)\underline{u}_4(t,r) - h\int_0^2 \frac{1}{4}t\underline{u}_4(s,r)\,ds = -\frac{1}{32}ht(r+1)(2+h)^4$$

$$\underline{u}_6(t,r) = (1+h)\underline{u}_5(t,r) - h\int_0^2 \frac{1}{4}t\underline{u}_5(s,r)\,ds = -\frac{1}{64}ht(r+1)(2+h)^5$$

and

$$\overline{u}_0(t,r) = \bar{0}$$

$$\overline{u}_1(t,r) = -h\underline{f}(t,r) = -\frac{1}{2}ht(3-r)$$

$$\overline{u}_2(t,r) = (1+h)\overline{u}_1(t,r) - h\int_0^2 \frac{1}{4}t.\overline{u}_1(s,r)\,ds = -\frac{1}{4}ht(3-r)(2+h)$$

$$\overline{u}_3(t,r) = (1+h)\overline{u}_2(t,r) - h\int_0^2 \frac{1}{4}t.\overline{u}_2(s,r)\,ds = -\frac{1}{8}ht(3-r)(2+h)^2$$

$$\overline{u}_4(t,r) = (1+h)\overline{u}_3(t,r) - h\int_0^2 \frac{1}{4}t\overline{u}_3(s,r)\,ds = -\frac{1}{16}ht(3-r)(2+h)^3$$

$$\overline{u}_5(t,r) = (1+h)\overline{u}_4(t,r) - h\int_0^2 \frac{1}{4}t\overline{u}_4(s,r)\,ds = -\frac{1}{32}ht(3-r)(2+h)^4$$

$$\overline{u}_6(t,r) = (1+h)\overline{u}_5(t,r) - h\int_0^2 \frac{1}{4}t\overline{u}_5(s,r)\,ds = -\frac{1}{64}ht(3-r)(2+h)^5$$

Then we approximate $\underline{g}(t,r)$ with

$$\begin{aligned}\underline{g}_6(t,r) &= \sum_{n=1}^{6}\underline{u}_n(t,r)\\ &= -ht(r+1)\left[\frac{1}{2}+\frac{1}{4}(2+h)+\frac{1}{8}(2+h)^2+\frac{1}{16}(2+h)^3\right.\\ &\quad\left.+\frac{1}{32}(2+h)^4+\frac{1}{64}(2+h)^5\right]\\ &= -ht(r+1).\sum_{n=1}^{6}\frac{1}{2^n}.(2+h)^{n-1}\end{aligned}$$

and $\overline{g}(t,r)$ with

$$\begin{aligned}\overline{g}_6(t,r) &= \sum_{n=1}^{6}\overline{u}_n(t,r)\\ &= -ht(3-r)\left[\frac{1}{2}+\frac{1}{4}(2+h)+\frac{1}{8}(2+h)^2+\frac{1}{16}(2+h)^3\right.\\ &\quad\left.+\frac{1}{32}(2+h)^4+\frac{1}{64}(2+h)^5\right]\\ &= -ht(3-r).\sum_{n=1}^{6}\frac{1}{2^n}.(2+h)^{n-1}\end{aligned}$$

3.3 Adomain Decomposition Method (ADM)

Since the beginning of 1980s the scientists and engineers did apply the Adomain decomposition method to functional equations in order to calculate the solutions as an infinite series which usually converges to the exact solution. However, the Adomain decomposition method is a special case of homotopy analysis method so we present the following theorem.

Theorem (3.10) [52]: If we set the convergence-controller parameter $h = -1$ in the frame of homotopy analysis method when it is applied on integral equations, the method will be converted to Adomain decomposition method.

Proof: see [52].

We will solve linear system (1.12). It can be written as follows [1]:

$$\underline{g}_i(t,r) = \underline{f_i}(t,r) + N_i\left(\underline{g}_1, \dots, \underline{g}_n\right)(t,r)$$

$$\overline{g}_i(t,r) = \overline{f}_i(t,r) + N_i(\overline{g}_1, \dots, \overline{g}_n)(t,r) \tag{3.21}$$

where

$$N_i\left(\underline{g}_1, \dots, \underline{g}_n\right)(t,r) = \int_a^b \sum_{j=1}^{n} K_{ij}(s,t)\underline{g}_j(s,r)\, ds$$

$$N_i(\overline{g}_1, \dots, \overline{g}_n)(t,r) = \int_a^b \sum_{j=1}^{n} K_{ij}(s,t)\overline{g}_j(s,r)\, ds \tag{3.22}$$

In order to use the Adomain decomposition method we need to represent

$g_i(t,r) = \left(\underline{g}_i(t,r), \overline{g}_i(t,r)\right)$ in a series form:

$$\underline{g}_i(t,r) = \sum_{m=0}^{\infty} \underline{g}_{im}(t,r)$$

$$\overline{g}_i(t,r) = \sum_{m=0}^{\infty} \overline{g}_{im}(t,r) \tag{3.23}$$

and letting

$$N_i\left(\underline{g}_1, \dots, \underline{g}_n\right)(t,r) = \sum_{m=0}^{\infty} \underline{A}_{im}$$

$$N_i\left(\overline{g}_1, \dots, \overline{g}_n\right)(t,r) = \sum_{m=0}^{\infty} \overline{A}_{im} \qquad , m = 0,1, \dots \tag{3.24}$$

where $A_{im} = \left(\underline{A}_{im}, \overline{A}_{im}\right)$ are Adomain polynomials.

Now, using equation (3.23) and (3.24), then equation (3.21) can be written as follows:

$$\sum_{m=0}^{\infty} \underline{g}_{im} = \underline{f}_i + \sum_{m=0}^{\infty} \underline{A}_{im}\left(\underline{g}_{10}, \dots, \underline{g}_{1m}, \dots, \underline{g}_{n0}, \dots, \underline{g}_{nm}\right)$$

$$\sum_{m=0}^{\infty} \overline{g}_{im} = \overline{f}_i + \sum_{m=0}^{\infty} \overline{A}_{im}\left(\overline{g}_{10}, \dots, \overline{g}_{1m}, \dots, \overline{g}_{n0}, \dots, \overline{g}_{nm}\right) \tag{3.25}$$

To obtain the Adomain's polynomial we introduce a parameter λ for convenience, so we have:

$$\underline{g}_{i\lambda}(t,r) = \sum_{m=0}^{\infty} \underline{g}_{im}(t,r)\,\lambda^m$$

$$\overline{g}_{i\lambda}(t,r) = \sum_{m=0}^{\infty} \overline{g}_{im}(t,r)\,\lambda^m \tag{3.26}$$

and

$$N_{i\lambda}\left(\underline{g}_{1\lambda}, \dots, \underline{g}_{n\lambda}\right) = \sum_{m=0}^{\infty} \underline{A}_{im}\,\lambda^m$$

$$N_{i\lambda}\left(\overline{g}_{1\lambda}, \dots, \overline{g}_{n\lambda}\right) = \sum_{m=0}^{\infty} \overline{A}_{im}\,\lambda^m \qquad , m = 0,1, \dots \tag{3.27}$$

then, we obtain the Adomain's polynomial $A_{im} = \left(\underline{A}_{im}, \overline{A}_{im}\right)$ as [26]:

$$\underline{A}_{im} = \frac{1}{m!}\left[\frac{d^m}{d\lambda^m} N_{i\lambda}\left(\underline{g}_{1\lambda}, \dots, \underline{g}_{n\lambda}\right)\right]_{\lambda=0}$$

$$\overline{A}_{im} = \frac{1}{m!}\left[\frac{d^m}{d\lambda^m} N_{i\lambda}\left(\overline{g}_{1\lambda}, \dots, \overline{g}_{n\lambda}\right)\right]_{\lambda=0} \tag{3.28}$$

from equations (3.26) and (3.27) , we get

$$\begin{aligned}\underline{A}_{im}&\left(\underline{g}_{10}, \dots, \underline{g}_{1m}, \dots, \underline{g}_{n0}, \dots, \underline{g}_{nm}\right)\\ &= \int_a^b \sum_{j=1}^{n} K_{ij}(s,t)\,\frac{1}{m!}\frac{d^m}{d\lambda^m}\sum_{l=0}^{\infty} \underline{g}_{jm}\,\lambda^l\bigg|_{\lambda=0} ds\\ &= \int_a^b \sum_{j=1}^{n} K_{ij}(s,t)\underline{g}_{jm}(s,r)\,ds\end{aligned}$$

$$\begin{aligned}\overline{A}_{im}&\left(\overline{g}_{10}, \dots, \overline{g}_{1m}, \dots, \overline{g}_{n0}, \dots, \overline{g}_{nm}\right)\\ &= \int_a^b \sum_{j=1}^{n} K_{ij}(s,t)\,\frac{1}{m!}\frac{d^m}{d\lambda^m}\sum_{l=0}^{\infty} \overline{g}_{jm}\,\lambda^l\bigg|_{\lambda=0} ds\\ &= \int_a^b \sum_{j=1}^{n} K_{ij}(s,t)\overline{g}_{jm}(s,r)\,ds\end{aligned} \tag{3.29}$$

Now, from equation (3.25) the solution of equation (1.12) will be as follows:

$$\underline{g}_{i0} = \underline{f}_i$$

$$\underline{g}_{i,m+1} = \int_a^b \sum_{j=1}^{n} K_{ij}(s,t)\underline{g}_{jm}(s,r)\,ds \tag{3.30}$$

and

$$\overline{g}_{i0} = \overline{f}_i$$

$$\overline{g}_{i,m+1} = \int_a^b \sum_{j=1}^{n} K_{ij}(s,t)\overline{g}_{jm}(s,r)\,ds \tag{3.31}$$

we usually approximate $g(t,r) = \left(\underline{g}(t,r), \overline{g}(t,r)\right)$ by [3]:

$$\underline{\phi}_{in} = \sum_{m=0}^{n-1} \underline{g}_{im}(t,r)$$

$$\overline{\phi}_{in} = \sum_{m=0}^{n-1} \overline{g}_{im}(t,r) \tag{3.32}$$

where

$$\lim_{n\to\infty} \underline{\phi}_{in} = \underline{g}_i(t,r)$$

$$\lim_{n\to\infty} \overline{\phi}_{in} = \overline{g}_i(t,r)$$

Example (3.3): Consider the fuzzy Fredholm integral equation (1.8) with

$$\underline{f}(t,r) = \frac{1}{2}(r+1).t$$

$$\overline{f}(t,r) = \frac{1}{2}(3-r).t$$

and kernel

$$k(s,t) = \frac{1}{4}t\,, \quad 0 \le s,t \le 2$$

on the interval [0,2]. Some of first terms of Adomian decomposition series are

$$\underline{u}_0(t,r) = \frac{1}{2}(r+1).t$$

$$\underline{u}_1(t,r) = \int_0^2 \frac{1}{4}t.\underline{u}_0(s,r)ds = \frac{1}{4}(r+1).t$$

$$\underline{u}_2(t,r) = \int_0^2 \frac{1}{4}t.\underline{u}_1(s,r)ds = \frac{1}{8}(r+1).t$$

$$\underline{u}_3(t,r) = \int_0^2 \frac{1}{4}t.\underline{u}_2(s,r)ds = \frac{1}{16}(r+1).t$$

$$\underline{u}_4(t,r) = \int_0^2 \frac{1}{4}t.\underline{u}_3(s,r)ds = \frac{1}{32}(r+1).t$$

$$\underline{u}_5(t,r) = \int_0^2 \frac{1}{4}t.\underline{u}_4(s,r)ds = \frac{1}{64}(r+1).t$$

$$\underline{u}_6(t,r) = \int_0^2 \frac{1}{4}t.\underline{u}_5(s,r)ds = \frac{1}{128}(r+1).t$$

and

$$\overline{u}_0(t,r) = \frac{1}{2}(3-r).t$$

$$\overline{u}_1(t,r) = \int_0^2 \frac{1}{4}t.\overline{u}_0(s,r)\,ds = \frac{1}{4}(3-r).t$$

$$\overline{u}_2(t,r) = \int_0^2 \frac{1}{4}t.\overline{u}_1(s,r)\,ds = \frac{1}{8}(3-r).t$$

$$\overline{u}_3(t,r) = \int_0^2 \frac{1}{4}t.\overline{u}_2(s,r)\,ds = \frac{1}{16}(3-r).t$$

$$\overline{u}_4(t,r) = \int_0^2 \frac{1}{4}t.\overline{u}_3(s,r)\,ds = \frac{1}{32}(3-r).t$$

$$\overline{u}_5(t,r) = \int_0^2 \frac{1}{4}t.\overline{u}_4(s,r)\,ds = \frac{1}{64}(3-r).t$$

$$\overline{u}_6(t,r) = \int_0^2 \frac{1}{4}t.\overline{u}_5(s,r)\,ds = \frac{1}{128}(3-r).t$$

then we approximate $\underline{g}(t,r)$ with

$$\underline{\phi}_7(t,r) = \frac{127}{128}t(r+1) = (1-2^{-7})t(r+1)$$

and

$$\overline{\phi}_7(t,r) = \frac{127}{128}t(3-r) = (1-2^{-7})t(3-r)$$

where

$$\underline{g}(t,r) = \lim_{n\to\infty}\underline{\phi}_n = \lim_{n\to\infty}(1-2^{-n})t(r+1) = (r+1).t$$

$$\overline{g}(t,r) = \lim_{n\to\infty}\overline{\phi}_n = \lim_{n\to\infty}(1-2^{-n})t(3-r) = (3-r).t$$

3.4 Fuzzy Differential Transformation Method (FDTM)

In this section, we are going to use fuzzy differential transformation method for solving fuzzy Fredholm integral equation of the second kind to obtain a series solutions.

Theorem (3.11) [6]: Consider the fuzzy-valued function

$g(t,r) = \left(\underline{g}(t,r), \overline{g}(t,r)\right)$, for $r \in [0,1]$, Then:

1) If g is (i)-differentiable, then $\underline{g}(t,r)$ and $\overline{g}(t,r)$ are differentiable functions and $g'(t,r) = \left(\underline{g}'(t,r), \overline{g}'(t,r)\right)$

2) If g is (ii)-differentiable, then $\underline{g}(t,r)$ and $\overline{g}(t,r)$ are differentiable functions and $g'(t,r) = \left(\overline{g}'(t,r), \underline{g}'(t,r)\right)$.

Definition (3.9) [12]: Let $g(t)$ be differentiable of order n in the time domain T, then:

If g is (i)-differentiable, $i.e. \underline{\varphi}(t,n,r) = \frac{d^n\left(\underline{g}(t,r)\right)}{dt^n}, \overline{\varphi}(t,n,r) = \frac{d^n\left(\overline{g}(t,r)\right)}{dt^n}$

then

$$\underline{G_i}(n,r) = \underline{\varphi}(t_i,n,r) = \left.\frac{d^n\left(\underline{g}(t,r)\right)}{dt^n}\right]_{t=t_i},$$

$$\overline{G}_i(n,r) = \overline{\varphi}(t_i,n,r) = \left.\frac{d^n\left(\overline{g}(t,r)\right)}{dt^n}\right]_{t=t_i}$$

and

$\forall t \in T,\ \forall n \in N, and\ r \in [0,1]$, and if g is (ii)-differentiable,

$$i.e. \overline{\varphi}(t,n,r) = \frac{d^n\left(\underline{g}(t,r)\right)}{dt^n}, then$$

$$\begin{cases} \underline{G_i}(n,r) = \overline{\varphi}(t_i,n,r) = \left.\dfrac{d^n\left(\overline{g}(t,r)\right)}{dt^n}\right]_{t=t_i} \\ \overline{G}_i(n,r) = \underline{\varphi}(t_i,n,r) = \left.\dfrac{d^n\left(\underline{g}(t,r)\right)}{dt^n}\right]_{t=t_i} \end{cases} , where\ n\ is\ odd$$

$$\begin{cases} \underline{G_i}(n,r) = \overline{\varphi}(t_i,n,r) = \left.\dfrac{d^n\left(\underline{g}(t,r)\right)}{dt^n}\right]_{t=t_i} \\ \overline{G}_i(n,r) = \underline{\varphi}(t_i,n,r) = \left.\dfrac{d^n\left(\overline{g}(t,r)\right)}{dt^n}\right]_{t=t_i} \end{cases} , where\ n\ is\ even$$

$G(t,r) = \left(\underline{G}(t,r), \overline{G}(t,r)\right)$ is called the spectrum at $t = t_i$ in the domain N.

If g is (i)-differentiable, then $g(t,r)$ can be represented as

$$\underline{g}(t,r) = \sum_{n=0}^{\infty} \frac{(t-t_i)^n}{n!} \underline{G}(n,r)$$

$$\overline{g}(t,r) = \sum_{n=0}^{\infty} \frac{(t-t_i)^n}{n!} \overline{G}(n,r)$$

and if g is (ii)-differentiable, then

$$\underline{g}(t,r) = \sum_{n=1,odd}^{\infty} \frac{(t-t_i)^n}{n!} \overline{G}(n,r) + \sum_{n=0,even}^{\infty} \frac{(t-t_i)^n}{n!} \underline{G}(n,r)$$

$$\overline{g}(t,r) = \sum_{n=1,odd}^{\infty} \frac{(t-t_i)^n}{n!}\underline{G}(n,r) + \sum_{n=0,even}^{\infty} \frac{(t-t_i)^n}{n!}\overline{G}(n,r)$$

The mentioned equations are known as the inverse transformation of $G(n,r)$. If $G(n,r)$ is (i)-differentiable then it is defined as follows

$$\underline{G}(n,r) = M(n)\left[\frac{d^n\left(q(t)\underline{g}(t,r)\right)}{dt^n}\right]_{t=0}$$

$$\overline{G}(n,r) = M(n)\left[\frac{d^n\left(q(t)\overline{g}(t,r)\right)}{dt^n}\right]_{t=0}$$

and if $G(n,r)$ is (i)-differentiable then it is defined as follows

$$\begin{cases} \underline{G}(n,r) = M(n)\left[\dfrac{d^n\left(q(t)\overline{g}(t,r)\right)}{dt^n}\right]_{t=0} \\ \overline{G}(n,r) = M(n)\left[\dfrac{d^n\left(q(t)\underline{g}(t,r)\right)}{dt^n}\right]_{t=0} \end{cases} \quad , where\ n\ is\ odd$$

$$\begin{cases} \underline{G}(n,r) = M(n)\left[\dfrac{d^n\left(q(t)\underline{g}(t,r)\right)}{dt^n}\right]_{t=0} \\ \overline{G}(n,r) = M(n)\left[\dfrac{d^n\left(q(t)\overline{g}(t,r)\right)}{dt^n}\right]_{t=0} \end{cases} \quad , where\ n\ is\ even$$

If g is (i)-differentiable, then $g(t,r)$ can be described as

$$\underline{g}(t,r) = \frac{1}{q(t)}\sum_{n=0}^{\infty} \frac{(t-t_i)^n}{n!}\frac{\underline{G}(n,r)}{M(n)}$$

$$\overline{g}(t,r) = \frac{1}{q(t)}\sum_{n=0}^{\infty} \frac{(t-t_i)^n}{n!}\frac{\overline{G}(n,r)}{M(n)}$$

and if g is (ii)-differentiable, then

$$\underline{g}(t,r)=\frac{1}{q(t)}\left(\sum_{n=1,odd}^{\infty}\frac{(t-t_i)^n}{n!}\frac{\overline{G}(n,r)}{M(n)}+\sum_{n=0,even}^{\infty}\frac{(t-t_i)^n}{n!}\frac{\underline{G}(n,r)}{M(n)}\right)$$

$$\overline{g}(t,r)=\frac{1}{q(t)}\left(\sum_{n=1,odd}^{\infty}\frac{(t-t_i)^n}{n!}\frac{\underline{G}(n,r)}{M(n)}+\sum_{n=0,even}^{\infty}\frac{(t-t_i)^n}{n!}\frac{\overline{G}(n,r)}{M(n)}\right)$$

where $M(n)>0$ and it is called the weighting factor, and $q(t)>0$ is regarded as kernel correspond to $g(t)$. Now, the transformation that is applied in this section is $M(n)=\frac{H^n}{n!}$ and $q(t)=1$, where H is the time horizon of interest.

If g is (i)-differentiable, then

$$\underline{G}(t,r)=\frac{H^n}{n!}\frac{d^n\underline{g}(t,r)}{dt^n}$$

$$\overline{G}(t,r)=\frac{H^n}{n!}\frac{d^n\overline{g}(t,r)}{dt^n}$$

and if g is (ii)-differentiable, then

$$\begin{cases}\underline{G}(t,r)=\dfrac{H^n}{n!}\dfrac{d^n\overline{g}(t,r)}{dt^n}\\ \overline{G}(t,r)=\dfrac{H^n}{n!}\dfrac{d^n\underline{g}(t,r)}{dt^n}\end{cases}\qquad ,where\ n\ is\ odd$$

$$\begin{cases}\underline{G}(t,r)=\dfrac{H^n}{n!}\dfrac{d^n\underline{g}(t,r)}{dt^n}\\ \overline{G}(t,r)=\dfrac{H^n}{n!}\dfrac{d^n\overline{g}(t,r)}{dt^n}\end{cases}\qquad ,where\ n\ is\ even$$

Using the differential transform, an integral equation in the domain of interest can be transformed to an algebraic equation in the N domain and

$g(t)$ can be obtained by finite-term Taylor series plus a reminder, so if $g(t)$ is (i)-differentiable it is obtained as follows

$$\underline{g}(t,r) = \frac{1}{q(t)}\sum_{n=0}^{\infty}\frac{(t-t_0)^n}{n!}\frac{\underline{G}(n,r)}{M(n)} + R_{n+1}(t)$$

$$= \sum_{n=0}^{\infty}\left(\frac{t-t_0}{H}\right)^n \underline{G}(n,r) + R_{n+1}(t)$$

$$\overline{g}(t,r) = \frac{1}{q(t)}\sum_{n=0}^{\infty}\frac{(t-t_0)^n}{n!}\frac{\overline{G}(n,r)}{M(n)} + R_{n+1}(t)$$

$$= \sum_{n=0}^{\infty}\left(\frac{t-t_0}{H}\right)^n \overline{G}(n,r) + R_{n+1}(t)$$

and if g is (ii)-differentiable, then

$$\underline{g}(t,r) = \frac{1}{q(t)}\left(\sum_{n=1,odd}^{\infty}\frac{(t-t_0)^n}{n!}\frac{\overline{G}(n,r)}{M(n)} + \sum_{n=0,even}^{\infty}\frac{(t-t_0)^n}{n!}\frac{\underline{G}(n,r)}{M(n)}\right)$$
$$+ R_{n+1}(t)$$

$$= \sum_{n=1,odd}^{\infty}\left(\frac{t-t_0}{H}\right)^n \overline{G}(n,r) + \sum_{n=0,even}^{\infty}\left(\frac{t-t_0}{H}\right)^n \underline{G}(n,r) + R_{n+1}(t)$$

$$\overline{g}(t,r) = \frac{1}{q(t)}\left(\sum_{n=1,odd}^{\infty}\frac{(t-t_0)^n}{n!}\frac{\underline{G}(n,r)}{M(n)} + \sum_{n=0,even}^{\infty}\frac{(t-t_0)^n}{n!}\frac{\overline{G}(n,r)}{M(n)}\right)$$
$$+ R_{n+1}(t)$$

$$= \sum_{n=1,odd}^{\infty}\left(\frac{t-t_0}{H}\right)^n \underline{G}(n,r) + \sum_{n=0,even}^{\infty}\left(\frac{t-t_0}{H}\right)^n \overline{G}(n,r) + R_{n+1}(t)$$

We want to find the solution of equation (1.8) at equally spaced points $a = t_0 < \cdots < t_m = b$, $t_i = a + ih$, $h = \frac{b-a}{M}$, $\forall i = 0, \dots, M$.

Now the domain of interest is divided to M sub-domains and the fuzzy approximation functions in each sub-domain are $g_i(t,r)$ for

$i = 0, \dots, M-1$ respectively.

Definition (3.10) [13]: Let $g(t,r)$ be differentiable fuzzy-valued function then the one-dimensional differential transform is defined as follows:

$$\underline{G}(n,r) = \frac{1}{n!}\left[\frac{d^n \underline{g}(t,r)}{dt^n}\right]_{t=0}$$

$$\overline{G}(n,r) = \frac{1}{n!}\left[\frac{d^n \overline{g}(t,r)}{dt^n}\right]_{t=0}$$

where $g(t,r)$ is the original function and $G(n,r)$ is the transformed function.

Definition (3.11): The differential inverse transform of $G(n,r)$ on the grid points (t_{i+1}) is defined by:

$$\underline{g}(t_{i+1},r) \approx \underline{g_i}(t_{i+1},r) = \underline{G_i}(0,r) + \cdots + \underline{G_i}(M,r)(t_{i+1}-t_i)^M$$

$$= \sum_{j=0}^{M} \underline{G_i}(j,r)h^j$$

$$\overline{g}(t_{i+1},r) \approx \overline{g}_i(t_{i+1},r) = \overline{G}_i(0,r) + \cdots + \overline{G}_i(M,r)(t_{i+1}-t_i)^M$$

$$= \sum_{j=0}^{M} \overline{G}_i(j,r)h^j.$$

Theorem (3.12) [21]: Let $g(t,r), l(t,r)$ and $f(t,r)$ are fuzzy-valued functions and the differential transformations are $G(n,r), L(n,r)$ and $F(n,r)$ respectively. Then:

1) If $g(t,r) = l(t,r) + f(t,r)$, then $G(n,r) = L(n,r) + F(n,r)$

2) If $g(t,r) = l(t,r) - f(t,r)$, then $G(n,r) = L(n,r) - F(n,r)$

3) If $g(t,r) = cf(t,r)$, then $G(n,r) = cF(n,r)$, where c is a constant.

Theorem (3.13) [21]: If $g(t,r) = x^p$, then $G(n,r) = \delta(n-p)$, where

$\delta(n-p) = \begin{cases} 1, & n = p \\ 0, & n \neq p \end{cases}.$

Theorem (3.14) [14]: Let $g(t,r)$ and $f(t,r)$ are fuzzy-valued functions and the differential transformations are $G(n,r)$ and $F(n,r)$ respectively. If

$f(t,r) = \int_{t_0}^{t} g(s,r)ds, r \in [0,1]$, then $F(n) = \frac{G(n-1)}{n}, n \geq 1$.

Proof: Using definition of fuzzy differential transform method, we get

$$\underline{f}(t,r) = \int_{t_0}^{t} \underline{g}(s,r)ds = \int_{t_0}^{t} \sum_{n=0}^{\infty} \underline{G}(n,r)(s-t_0)^n ds$$

$$= \sum_{n=0}^{\infty} \frac{\underline{G}(n,r)}{n+1}(s-t_0)^{n+1}\Bigg]_{t_0}^{t} = \sum_{n=0}^{\infty} \frac{\underline{G}(n,r)}{n+1}(t-t_0)^{n+1}$$

and

$$\overline{f}(t,r) = \int_{t_0}^{t} \overline{g}(s,r)ds = \int_{t_0}^{t} \sum_{n=0}^{\infty} \overline{G}(n,r)(s-t_0)^n ds$$

$$= \sum_{n=0}^{\infty} \frac{\overline{G}(n,r)}{n+1}(s-t_0)^{n+1}\Bigg]_{t_0}^{t} = \sum_{n=0}^{\infty} \frac{\overline{G}(n,r)}{n+1}(t-t_0)^{n+1}$$

we shift the index from $n = 0$ to $n = 1$, then we obtain

$$\underline{f}(t,r) = \sum_{n=0}^{\infty} \frac{\underline{G}(n-1,r)}{n}(t-t_0)^{n}$$

and

$$\overline{f}(t,r) = \sum_{n=0}^{\infty} \frac{\overline{G}(n-1,r)}{n}(t-t_0)^{n}$$

Finally, using the definition of fuzzy differential transform method, we get

$$F(n,r) = \frac{G(n-1,r)}{n}, r \in [0,1], n \geq 1.$$

Theorem (3.15) [21]: Let $U(n,r), F(n,r)$ and $G(n,r)$ be differential transformations of the positive real-valued function $u(t)$ and the fuzzy-valued functions $f(t,r)$ and $g(t,r)$ respectively. If

$f(t,r) = \int_{t_0}^{t} u(s)g(s,r)ds$, then we have

$$\underline{F}(n,r) = \frac{1}{n}\sum_{l=0}^{n-1} U(l)\underline{G}(n-l-1,r)$$

and

$$\overline{F}(n,r) = \frac{1}{n}\sum_{l=0}^{n-1} U(l)\overline{G}(n-l-1,r)$$

Proof: Using the definition of fuzzy differential transform method, we have

$$F(0,r)=\frac{1}{0!}\left[\int_{t_0}^{t}u(s)g(s,r)ds\right]_{t=t_0}=\bar{0}$$

$$F(1,r)=\frac{1}{1!}\frac{d}{dt}\left[\int_{t_0}^{t}u(s)g(s,r)ds\right]_{t=t_0}$$

$$=\left[\frac{d}{dt}\int_{t_0}^{t}u(s)\underline{g}(s,r)ds\,,\frac{d}{dt}\int_{t_0}^{t}u(s)\overline{g}(s,r)ds\right]_{t=t_0}$$

$$=\left[u(t)\underline{g}(t,r),u(t)\overline{g}(t,r)\right]_{t=t_0}=U(0)G(0,r)$$

$$F(2,r)=\frac{1}{2!}\frac{d^2}{dt^2}\left[\int_{t_0}^{t}u(s)g(s,r)ds\right]_{t=t_0}$$

$$=\frac{1}{2!}\left[u'(t)\,g(t,r)+u(t)\,g'(t,r)\right]_{t=t_0}$$

$$=\frac{1}{2}[U(1)G(0,r)+U(0)G(1,r)]_{t=t_0}$$

$$F(3,r)=\frac{1}{3!}\frac{d^3}{dt^3}\left[\int_{t_0}^{t}u(s)g(s,r)ds\right]_{t=t_0}$$

$$=\frac{1}{3!}\left[u''(t)\,g(t,r)+2\,u'(t)\;g'(t,r)\;+u(t)\,g''(t,r)\right]_{t=t_0}$$

$$=\frac{1}{3!}[U(2)G(0,r)++U(1)G(1,r)+U(0)G(2,r)]_{t=t_0}$$

In general, we get

$$\underline{F}(n,r)=\frac{1}{n}\sum_{l=0}^{n-1}U(l)\underline{G}(n-l-1,r)$$

and

$$\overline{F}(n,r)=\frac{1}{n}\sum_{l=0}^{n-1}U(l)\overline{G}(n-l-1,r)$$

Theorem (3.16) [14]: Let $U(n,r), F(n,r)$ and $G(n,r)$ be differential transformations of the positive real-valued function $u(t)$ and the fuzzy-valued functions $f(t,r)$ and $g(t,r)$ respectively. If

$f(t,r) = u(t)\int_{t_0}^{t} g(s,r)ds$, then we have

$$\underline{F}(n,r) = \sum_{l=1}^{n} \frac{1}{l} U(n-l)\underline{G}(l-1,r)$$

and

$$\overline{F}(n,r) = \sum_{l=1}^{n} \frac{1}{l} U(n-l)\overline{G}(l-1,r)$$

Proof: Using the definition of fuzzy differential transform method, we have

$$F(0,r) = \frac{1}{0!}\left[u(t)\int_{t_0}^{t} g(s,r)ds\right]_{t=t_0} = \bar{0}$$

$$F(1,r) = \frac{1}{1!}\frac{d}{dt}\left[u(t)\int_{t_0}^{t} g(s,r)ds\right]_{t=t_0}$$

$$= \left[\frac{d}{dt}\big(u(t)\big).\int_{t_0}^{t} \underline{g}(s,r)ds\,, u(t).\frac{d}{dt}\int_{t_0}^{t} \overline{g}(s,r)ds\right]_{t=t_0}$$

$$= \left[u'\big(t\big)\,\underline{g}(t,r), u(t)\overline{g}(t,r)\right]_{t=t_0} = U(0)G(0,r)$$

$$F(2,r) = \frac{1}{2!}\frac{d^2}{dt^2}\left[u(t)\int_{t_0}^{t} g(s,r)ds\right]_{t=t_0}$$

$$= \frac{1}{2!}\left[u''(t).\int_{t_0}^{t} g(s,r)ds + 2\,u'\big(t\big)\,g(t,r) + u(t)\,g'\big(t,r\big)\right]_{t=t_0}$$

$$= \left[U(1)G(0,r) + \frac{1}{2}U(0)G(1,r)\right]_{t=t_0}$$

In general, we get

$$\underline{F}(n,r) = \sum_{l=1}^{n} \frac{1}{l} U(n-l)\underline{G}(l-1,r)$$

and

$$\overline{F}(n,r) = \sum_{l=1}^{n} \frac{1}{l} U(n-l)\overline{G}(l-1,r)$$

Taking the fuzzy differential transformation of system (1.8), then we get

$$\underline{G}(n,r) = \underline{F}(n,r) + \frac{\lambda}{n}\sum_{l=0}^{n-1} U(l)\underline{G}(n-l-1,r)$$

$$\overline{G}(n,r) = \overline{F}(n,r) + \frac{\lambda}{n}\sum_{l=0}^{n-1} U(l)\overline{G}(n-1-1,r)$$

where $\lambda > 0$, and $G(n,r), F(n,r)$ and $U(n)$ are fuzzy differential transformation of $g(t,r), f(t,r)$ which are fuzzy-valued functions and $u(t)$ is positive real-valued function. And we have $\underline{G}(0,r) = \underline{F}(0,r)$ and

$\overline{G}(0,r) = \overline{F}(0,r)$.

Example (3.4): Let us consider the fuzzy Fredholm integral equation (1.8) with

$$\underline{f}(t,r) = \frac{1}{2}(r+1).t$$

$$\overline{f}(t,r) = \frac{1}{2}(3-r).t$$

and kernel

$$k(s,t) = \frac{1}{4}t\,, \quad 0 \leq s,t \leq 2$$

on the interval [0,2]. Then applying fuzzy differential transform, we have

$$\underline{G}(n,r) = \frac{1}{2}(r+1)\delta(n-1) + \frac{1}{4}\sum_{l=1}^{n}\frac{1}{l}\delta(n-l-1)\underline{G}(l-1,r)$$

$$\overline{G}(n,r) = \frac{1}{2}(3-r)\delta(n-1) + \frac{1}{4}\sum_{l=1}^{n}\frac{1}{l}\delta(n-l-1)\overline{G}(l-1,r)$$

where $\underline{G}(0,r) = \overline{G}(0,r) = \bar{0}$. Consequently, we obtain,

$$\underline{G}(1,r) = \frac{1}{2}(r+1)\delta(1-1) + \frac{1}{4}\sum_{l=1}^{1}\frac{1}{l}\delta(1-l-1)\underline{G}(l-1,r) = \frac{1}{2}(r+1)$$

$$\underline{G}(2,r) = \frac{1}{2}(r+1)\delta(2-1) + \frac{1}{4}\sum_{l=1}^{2}\frac{1}{l}\delta(2-l-1)\underline{G}(l-1,r) = \bar{0}$$

$$\begin{aligned}\underline{G}(3,r) &= \frac{1}{2}(r+1)\delta(3-1) + \frac{1}{4}\sum_{l=1}^{3}\frac{1}{l}\delta(3-l-1)\underline{G}(l-1,r)\\ &= \frac{1}{2^4}(r+1)\end{aligned}$$

$$\underline{G}(4,r) = \frac{1}{2}(r+1)\delta(4-1) + \frac{1}{4}\sum_{l=1}^{4}\frac{1}{l}\delta(2-l-1)\underline{G}(l-1,r) = \bar{0}$$

$$\begin{aligned}\underline{G}(5,r) &= \frac{1}{2}(r+1)\delta(5-1) + \frac{1}{4}\sum_{l=1}^{5}\frac{1}{l}\delta(5-l-1)\underline{G}(l-1,r)\\ &= \frac{1}{2^8}(r+1)\end{aligned}$$

$$\underline{G}(6,r) = \frac{1}{2}(r+1)\delta(6-1) + \frac{1}{4}\sum_{l=1}^{6}\frac{1}{l}\delta(6-l-1)\underline{G}(l-1,r) = \bar{0}$$

$$\underline{G}(7,r) = \frac{1}{2}(r+1)\delta(7-1) + \frac{1}{4}\sum_{l=1}^{7}\frac{1}{l}\delta(7-l-1)\underline{G}(l-1,r)$$
$$= \frac{1}{3.\,2^{11}}(r+1)$$

$$\underline{G}(8,r) = \frac{1}{2}(r+1)\delta(8-1) + \frac{1}{4}\sum_{l=1}^{8}\frac{1}{l}\delta(8-l-1)\underline{G}(l-1,r) = \bar{0}$$

$$\underline{G}(9,r) = \frac{1}{2}(r+1)\delta(9-1) + \frac{1}{4}\sum_{l=1}^{9}\frac{1}{l}\delta(9-l-1)\underline{G}(l-1,r)$$
$$= \frac{1}{3.\,2^{15}}(r+1)$$

$$\underline{G}(10,r) = \frac{1}{2}(r+1)\delta(10-1) + \frac{1}{4}\sum_{l=1}^{10}\frac{1}{l}\delta(10-l-1)\underline{G}(l-1,r) = \bar{0}$$

$$\underline{G}(11,r) = \frac{1}{2}(r+1)\delta(11-1) + \frac{1}{4}\sum_{l=1}^{11}\frac{1}{l}\delta(11-l-1)\underline{G}(l-1,r)$$
$$= \frac{1}{3.5.\,2^{18}}(r+1)$$

and

$$\overline{G}(1,r) = \frac{1}{2}(3-r)\delta(1-1) + \frac{1}{4}\sum_{l=1}^{1}\frac{1}{l}\delta(1-l-1)\overline{G}(l-1,r) = \frac{1}{2}(3-r)$$

$$\overline{G}(2,r) = \frac{1}{2}(3-r)\delta(2-1) + \frac{1}{4}\sum_{l=1}^{2}\frac{1}{l}\delta(2-l-1)\overline{G}(l-1,r) = \bar{0}$$

$$\overline{G}(3,r) = \frac{1}{2}(3-r)\delta(3-1) + \frac{1}{4}\sum_{l=1}^{3}\frac{1}{l}\delta(3-l-1)\overline{G}(l-1,r)$$
$$= \frac{1}{2^{4}}(3-r)$$

$$\overline{G}(4,r) = \frac{1}{2}(3-r)\delta(4-1) + \frac{1}{4}\sum_{l=1}^{4}\frac{1}{l}\delta(4-l-1)\overline{G}(l-1,r) = \bar{0}$$

$$\overline{G}(5,r) = \frac{1}{2}(3-r)\delta(5-1) + \frac{1}{4}\sum_{l=1}^{4}\frac{1}{l}\delta(5-l-1)\overline{G}(l-1,r)$$
$$= \frac{1}{2^8}(3-r)$$

$$\overline{G}(6,r) = \frac{1}{2}(3-r)\delta(6-1) + \frac{1}{4}\sum_{l=1}^{6}\frac{1}{l}\delta(6-l-1)\overline{G}(l-1,r) = \bar{0}$$

$$\overline{G}(7,r) = \frac{1}{2}(3-r)\delta(7-1) + \frac{1}{4}\sum_{l=1}^{7}\frac{1}{l}\delta(7-l-1)\overline{G}(l-1,r)$$
$$= \frac{1}{3.\,2^{11}}(3-r)$$

$$\overline{G}(8,r) = \frac{1}{2}(3-r)\delta(8-1) + \frac{1}{4}\sum_{l=1}^{8}\frac{1}{l}\delta(8-l-1)\overline{G}(l-1,r) = \bar{0}$$

$$\overline{G}(9,r) = \frac{1}{2}(3-r)\delta(9-1) + \frac{1}{4}\sum_{l=1}^{9}\frac{1}{l}\delta(9-l-1)\overline{G}(l-1,r)$$
$$= \frac{1}{3.\,2^{15}}(3-r)$$

$$\overline{G}(10,r) = \frac{1}{2}(3-r)\delta(10-1) + \frac{1}{4}\sum_{l=1}^{10}\frac{1}{l}\delta(10-l-1)\overline{G}(l-1,r) = \bar{0}$$

$$\overline{G}(11,r) = \frac{1}{2}(3-r)\delta(11-1) + \frac{1}{4}\sum_{l=1}^{11}\frac{1}{l}\delta(11-l-1)\overline{G}(l-1,r)$$
$$= \frac{1}{3.5.\,2^{18}}(3-r)$$

Now, the approximate solution

$$\underline{g}(t,r) = \sum_{n=0}^{\infty}\underline{G}(n,r).\,t^n$$
$$= t(r+1)(\frac{1}{2} + \frac{1}{2^3}t^2 + \frac{1}{2^7}t^4 + \frac{1}{3.\,2^{11}}t^6 + \frac{1}{3.\,2^{15}}t^8$$
$$+ \frac{1}{3.5.\,2^{18}}t^{10} + \cdots)$$

and

$$\overline{g}(t,r) = \sum_{n=0}^{\infty} \overline{G}(n,r).t^n$$
$$= t(r+1)\left(\frac{1}{2} + \frac{1}{2^3}t^2 + \frac{1}{2^7}t^4 + \frac{1}{3.2^{11}}t^6 + \frac{1}{3.2^{15}}t^8 + \frac{1}{3.5.2^{18}}t^{10} + \cdots\right).$$

Chapter Four

Numerical Methods for Solving Linear Fuzzy Fredholm Integral Equation of the Second Kind

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Numerical Methods for Solving Linear Fuzzy Fredholm Integral Equation of the Second Kind

4.1 Taylor Expansion Method

We will use Taylor expansion method to solve linear fuzzy Fredholm integral equation of the second kind. This method is based on differentiating p-times both sides of the linear fuzzy Fredholm integral equation of the second kind and then substitute the Taylor series expansion for the unknown function. As a result, we obtain a linear system for which the solution of this system yields the unknown Taylor coefficients of the solution functions.

Now, we recall system (1.12) then we assume that

$$\begin{cases} \lambda_{ij}K_{i,j}(s,t) \geq 0\,, & a \leq s \leq c_{i,j} \\ \lambda_{ij}K_{i,j}(s,t) < 0\,, & c_{i,j} \leq s \leq b \end{cases}$$

The system (1.12) is transformed using the above assumption, that is,

$$\underline{g}_i(t,r) = \underline{f_i}(t,r) + \sum_{j=1}^{m} \lambda_{i,j} \left(\int_a^c k_{i,j}(s,t)\underline{g}_j(s,r)ds + \int_c^b k_{i,j}(s,t)\overline{g}_j(s,r)ds \right)$$

$$\overline{g}_i(t,r) = \overline{f}_i(t,r) + \sum_{j=1}^{m} \lambda_{i,j} \left(\int_a^c k_{i,j}(s,t)\overline{g}_j(s,r)ds + \int_c^b k_{i,j}(s,t)\underline{g}_j(s,r)ds \right)$$

$$i,j = 1,\dots,m \tag{4.1}$$

We differentiate both sides of each equation of the system (4.1) with respect to t, N times

$$\frac{\partial^{(p)}\underline{g}_i(t,r)}{\partial t^p}=\frac{\partial^{(p)}\underline{f}_i(t,r)}{\partial t^p}+\sum_{j=1}^{m}\lambda_{i,j}\left(\int_a^c\frac{\partial^{(p)}k_{i,j}(s,t)}{\partial t^p}\underline{g}_j(s,r)ds+\int_c^b\frac{\partial^{(p)}k_{i,j}(s,t)}{\partial t^p}\overline{g}_j(s,r)ds\right)\qquad i,j=1,\dots,m$$

$$\frac{\partial^{(p)}\overline{g}_i(t,r)}{\partial t^p}=\frac{\partial^{(p)}\overline{f}_i(t,r)}{\partial t^p}+\sum_{j=1}^{m}\lambda_{i,j}\left(\int_a^c\frac{\partial^{(p)}k_{i,j}(s,t)}{\partial t^p}\overline{g}_j(s,r)ds+\int_c^b\frac{\partial^{(p)}k_{i,j}(s,t)}{\partial t^p}\underline{g}_j(s,r)ds\right)\qquad p=0,\dots,N. \tag{4.2}$$

Using the following notations for abbreviation:

$$\underline{g}_i^{(p)}(z,r)=\left.\frac{\partial^{(p)}\underline{g}_i(t,r)}{\partial t^p}\right|_{t=z},\ \overline{g}_i^{(p)}(z,r)=\left.\frac{\partial^{(p)}\overline{g}_i(t,r)}{\partial t^p}\right|_{t=z}$$

$$\underline{f}_i^{(p)}(z,r)=\left.\frac{\partial^{(p)}\underline{f}_i(t,r)}{\partial t^p}\right|_{t=z},\ \overline{f}_i^{(p)}(z,r)=\left.\frac{\partial^{(p)}\overline{f}_i(t,r)}{\partial t^p}\right|_{t=z}$$

$$k_{i,j}^{(p)}(s,z)=\left.\frac{\partial^{(p)}k_{i,j}(s,t)}{\partial t^p}\right|_{t=z}\qquad ,i,j=1,\dots,m \tag{4.3}$$

Now, we expand the unknown functions $\underline{g}_i(s,r)$ and $\overline{g}_i(s,r)$ in Taylor series for multivariate variable at arbitrary point z and we neglect the truncation error:

$$\underline{g}_{j,N}(s,r) = \sum_{q=0}^{N} \frac{1}{q!} \underline{g}_i^{(p)}(s,r)(s-z)^p$$

$$\overline{g}_{j,N}(s,r) = \sum_{q=0}^{N} \frac{1}{q!} \overline{g}_i^{(p)}(s,r)\,(s-z)^p, a \le z \le b\,, j = 1, \dots, m \qquad (4.4)$$

Substituting equations (4.4) and (4.3) into equation (4.2), yields:

$$\begin{aligned}\underline{g}_i^{(p)}(z,r) &= \underline{f}_i^{(p)}(z,r) \\ &+ \sum_{j=1}^{m} \left(\sum_{q=0}^{N} \frac{\lambda_{i,j}}{q!} \int_a^c k_{i,j}^{(p)}(s,z)\underline{g}_j(z,r).\,(s-z)^p ds \right. \\ &\left. + \sum_{q=0}^{N} \frac{\lambda_{i,j}}{q!} \int_c^b k_{i,j}^{(p)}(s,z)\overline{g}_j(z,r).\,(s-z)^p ds \right)\end{aligned}$$

$$\begin{aligned}\overline{g}_i^{(p)}(z,r) &= \overline{f}_i^{(p)}(z,r) \\ &+ \sum_{j=1}^{m} \left(\sum_{q=0}^{N} \frac{\lambda_{i,j}}{q!} \int_a^c k_{i,j}^{(p)}(s,z)\overline{g}_j(z,r).\,(s-z)^p ds \right. \\ &\left. + \sum_{q=0}^{N} \frac{\lambda_{i,j}}{q!} \int_c^b k_{i,j}^{(p)}(s,z)\underline{g}_j(z,r).\,(s-z)^p ds \right) \qquad (4.5)\end{aligned}$$

Using the notation:

$$w_{p,q}^{(i,j)} = \frac{\lambda_{i,j}}{q!} \int_a^{c_{i,j}} k_{i,j}^{(p)}(s,z).\,(s-z)^p ds \qquad i,j = 1, \dots, m$$

$$w_{p,q}^{'(i,j)} = \frac{\lambda_{i,j}}{q!} \int_{c_{i,j}}^{b} k_{i,j}^{(p)}(s,z).\,(s-z)^p ds \qquad p,q = 0, \dots, N \qquad (4.6)$$

the equation (4.5) can be written as:

$$\underline{g}_i^{(p)}(z,r) = \underline{f}_i^{(p)}(z,r) + \sum_{j=1}^{m}\left(\sum_{q=0}^{N} w_{p,q}^{(i,j)}\,\underline{g}_j(z,r) + \sum_{q=0}^{N} w_{p,q}^{\prime(i,j)}\overline{g}_j(z,r)\right)$$

$$\overline{g}_i^{(p)}(z,r) = \overline{f}_i^{(p)}(z,r) + \sum_{j=1}^{m}\left(\sum_{q=0}^{N} w_{p,q}^{(i,j)}\overline{g}_j(z,r) + \sum_{q=0}^{N} w_{p,q}^{\prime(i,j)}\,\underline{g}_j(z,r)\right) \quad (4.7)$$

We can rearrange (4.7) as follow:

$$-\underline{f}_i^{(p)}(z,r) = -\underline{g}_i^{(p)}(z,r) + \sum_{j=1}^{m}\left(\sum_{q=0}^{N} w_{p,q}^{(i,j)}\,\underline{g}_j(z,r) + \sum_{q=0}^{N} w_{p,q}^{\prime(i,j)}\overline{g}_j(z,r)\right)$$

$$-\overline{f}_i^{(p)}(z,r) = -\overline{g}_i^{(p)}(z,r) + \sum_{j=1}^{m}\left(\sum_{q=0}^{N} w_{p,q}^{(i,j)}\overline{g}_j(z,r) + \sum_{q=0}^{N} w_{p,q}^{\prime(i,j)}\,\underline{g}_j(z,r)\right)$$

(4.8)

But when $p = q$, then we (4.8) becomes :

$$-\underline{f}_i^{(p)}(z,r) = \sum_{j=1}^{m}\sum_{q=0}^{N}\left(w_{p,q}^{(i,j)} - 1\right)\underline{g}_j(z,r)$$

$$-\overline{f}_i^{(p)}(z,r) = \sum_{j=1}^{m}\sum_{q=0}^{N}\left(w_{p,q}^{(i,j)} - 1\right)\overline{g}_j(z,r)$$

Equation (4.7) can then be written in the matrix form

$$WG = F \quad (4.9)$$

where

$$G=\begin{bmatrix}\underline{g}_1(z,r)\\ \vdots\\ \underline{g}_1^{(N)}(z,r)\\ \overline{g}_1(z,r)\\ \vdots\\ \overline{g}_1^{(N)}(z,r)\\ \vdots\\ \underline{g}_m(z,r)\\ \vdots\\ \underline{g}_m^{(N)}(z,r)\\ \overline{g}_m(z,r)\\ \vdots\\ \overline{g}_m^{(N)}(z,r)\end{bmatrix},F=\begin{bmatrix}-\underline{f}_1(z,r)\\ \vdots\\ -\underline{f}_1^{(N)}(z,r)\\ -\overline{f}_1(z,r)\\ \vdots\\ -\overline{f}_1^{(N)}(z,r)\\ \vdots\\ -\underline{f}_m(z,r)\\ \vdots\\ -\underline{f}_m^{(N)}(z,r)\\ -\overline{f}_m(z,r)\\ \vdots\\ -\overline{f}_m^{(N)}(z,r)\end{bmatrix},W=\begin{bmatrix}W^{(1,1)} & \cdots & W^{(1,m)}\\ \vdots & \ddots & \vdots\\ W^{(m,1)} & \cdots & W^{(m,m)}\end{bmatrix}$$

Parochial matrices $W^{(i,j)}$ are defined with the following elements [24]:

$$W^{(i,j)}=\begin{bmatrix}W_{1,1}^{(i,j)} & W_{1,2}^{(i,j)}\\ W_{2,1}^{(i,j)} & W_{2,2}^{(i,j)}\end{bmatrix}$$

where

$$W_{1,1}^{(i,j)}=W_{2,2}^{(i,j)}=\begin{bmatrix}w_{0,0}^{(i,j)}-1 & w_{0,1}^{(i,j)} & \dots & w_{0,N-1}^{(i,j)} & w_{0,N}^{(i,j)}\\ w_{1,0}^{(i,j)} & w_{1,1}^{(i,j)}-1 & \dots & w_{1,N-1}^{(i,j)} & w_{1,N}^{(i,j)}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ w_{N-1,0}^{(i,j)} & w_{N-1,1}^{(i,j)} & \dots & w_{N-1,N-1}^{(i,j)}-1 & w_{N-1,N}^{(i,j)}\\ w_{N,0}^{(i,j)} & w_{N,1}^{(i,j)} & \dots & w_{N,N-1}^{(i,j)} & w_{N,N}^{(i,j)}-1\end{bmatrix}$$

$$W_{1,2}^{\prime(i,j)}=W_{2,1}^{\prime(i,j)}=\begin{bmatrix}w_{0,0}^{\prime(i,j)} & w_{0,1}^{\prime(i,j)} & \dots & w_{0,N-1}^{\prime(i,j)} & w_{0,N}^{\prime(i,j)}\\ w_{1,0}^{\prime(i,j)} & w_{1,1}^{\prime(i,j)} & \dots & w_{1,N-1}^{\prime(i,j)} & w_{1,N}^{\prime(i,j)}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ w_{N-1,0}^{\prime(i,j)} & w_{N-1,1}^{\prime(i,j)} & \dots & w_{N-1,N-1}^{\prime(i,j)} & w_{N-1,N}^{\prime(i,j)}\\ w_{N,0}^{\prime(i,j)} & w_{N,1}^{\prime(i,j)} & \dots & w_{N,N-1}^{\prime(i,j)} & w_{N,N}^{\prime(i,j)}\end{bmatrix}$$

The aim of this method is to determine the coefficients of $\underline{g}_j(t,r)$ and $\overline{g}_j(t,r)$ in system (4.2).

Finally, we want to obtain the solution of system (4.9) as follow:

$$\underline{g}(t,r) = \sum_{p=0}^{N} \frac{1}{p!} \frac{\partial^{(p)} \underline{g}(z,r)}{\partial t^p}\bigg|_{t=z} (t-z)^p$$

$$\overline{g}(t,r) = \sum_{p=0}^{N} \frac{1}{p!} \frac{\partial^{(p)} \overline{g}(z,r)}{\partial t^p}\bigg|_{t=z} (t-z)^p, a \le z \le b\ , j = 1, \dots, m \qquad (4.10)$$

4.1.1 Convergence Analysis

In order to show the efficiency of the Taylor expansion method, we will prove that the approximate solution converges to the exact solution of system (1.12).

Theorem (4.1) [24]: If $\underline{g}_{j,N}(t,r)$ and $\overline{g}_{j,N}(t,r)$ are Taylor polynomials of degree N and their coefficients have been found by solving the linear system (4.9), then they converges to the exact solution of system (1.12), when $N \to \infty$.

Proof: Consider the fuzzy system (1.12). The series (4.4) converges to $\underline{g}_j(t,r)$ and $\overline{g}_j(t,r)$ respectively, $i.e.$

$$\underline{g}_j(t,r) = \lim_{N\to\infty} \underline{g}_{j,N}(t,r)$$

$$\overline{g}_j(t,r) = \lim_{N\to\infty} \overline{g}_{j,N}(t,r).$$

then we conclude that :

$$\underline{g}_{iN}(t,r) = \underline{f}_i(t,r) + \sum_{j=1}^{m} \lambda_{i,j}\left(\int_a^{c_{i,j}} k_{i,j}(s,t)\underline{g}_{jN}(s,r)ds + \int_{c_{i,j}}^{b} k_{i,j}(s,t)\overline{g}_{jN}(s,r)ds\right)$$

$$\overline{g}_{iN}(t,r) = \overline{f}_i(t,r) + \sum_{j=1}^{m} \lambda_{i,j}\left(\int_a^{c_{i,j}} k_{i,j}(s,t)\overline{g}_{jN}(s,r)ds + \int_{c_{i,j}}^{b} k_{i,j}(s,t)\underline{g}_{jN}(s,r)ds\right)$$

$$i,j = 1,\dots,m \tag{4.11}$$

Now, the error function $e_N(t,r) = \sum_{i=1}^{m} e_{i,N}(t,r)$

where $e_{i,N}(t,r) = \underline{e}_{i,N}(t,r) + \overline{e}_{i,N}(t,r)$, then we define the error function as a difference between system (4.1) and (4.11) as follows:

$$\underline{e}_{iN}(t,r) = \left(\underline{g}_i(t,r) - \underline{g}_{iN}(t,r)\right) + \sum_{j=1}^{m} \lambda_{i,j}\left(\int_a^{c_{i,j}} k_{i,j}(s,t)\left(\underline{g}_j(s,r) - \underline{g}_{jN}(s,r)\right)ds + \int_{c_{i,j}}^{b} k_{i,j}(s,t)\left(\underline{g}_j(s,r) - \underline{g}_{jN}(s,r)\right)ds\right)$$

and

$$\overline{e}_{i,N}(t,r) = \left(\overline{g}_i(t,r) - \overline{g}_{iN}(t,r)\right) + \sum_{j=1}^{m} \lambda_{i,j}\left(\int_a^{c_{i,j}} k_{i,j}(s,t)\left(\overline{g}_j(s,r) - \overline{g}_{iN}(s,r)\right)ds + \int_{c_{i,j}}^{b} k_{i,j}(s,t)\left(\overline{g}_j(s,r) - \overline{g}_{iN}(s,r)\right)ds\right)$$

Now,

$$\|e_N\| = \left\|\sum_{i=1}^{m} e_{iN}\right\| \leq \sum_{i=1}^{m}\|e_{iN}\| = \sum_{i=1}^{m}\|\underline{e}_{iN} + \overline{e}_{iN}\| \leq \sum_{i=1}^{m}\left(\|\underline{e}_{iN}\| + \|\overline{e}_{iN}\|\right)$$

$$= \sum_{i=1}^{m}\Bigg[\Bigg\|\Big(\underline{g}_i(t,r) - \underline{g}_{iN}(t,r)\Big)$$

$$+ \sum_{j=1}^{m} \lambda_{i,j}\left(\int_a^{c_{i,j}} k_{i,j}\left(\underline{g}_j(s,r) - \underline{g}_{jN}(s,r)\right) ds\right.$$

$$\left. + \int_{c_{i,j}}^{b} k_{i,j}\left(\underline{g}_j(s,r) - \underline{g}_{jN(s,r)}\right) ds\right)\Bigg\|$$

$$+ \Bigg\|\Big(\overline{g}_i(t,r) - \overline{g}_{iN}(t,r)\Big)$$

$$+ \sum_{j=1}^{m} \lambda_{i,j}\left(\int_a^{c_{i,j}} k_{i,j}\left(\overline{g}_j(s,r) - \overline{g}_{iN}(s,r)\right) ds\right.$$

$$\left. + \int_{c_{i,j}}^{b} k_{i,j}\left(\overline{g}_j(s,r) - \overline{g}_{iN}(s,r)\right) ds\right)\Bigg\|\Bigg]$$

$$\leq \sum_{i=1}^{m} \left\|\left(\underline{g}_i(t,r) - \underline{g}_{iN}(t,r)\right) + \left(\overline{g}_i(t,r) - \overline{g}_{iN}(t,r)\right)\right\|$$

$$+ \sum_{i=1}^{m}\Bigg\|\sum_{j=1}^{m} \lambda_{i,j}\left(\int_a^{c_{i,j}} k_{i,j}\left(\underline{g}_j(s,r) - \underline{g}_{jN}(s,r)\right) ds\right.$$

$$\left. + \int_{c_{i,j}}^{b} k_{i,j}\left(\underline{g}_j(s,r) - \underline{g}_{jN}(s,r)\right) ds\right)$$

$$+ \sum_{j=1}^{m} \lambda_{i,j}\left(\int_a^{c_{i,j}} k_{i,j}\left(\overline{g}_j(s,r) - \overline{g}_{iN}(s,r)\right) ds\right.$$

$$\left. + \int_{c_{i,j}}^{b} k_{i,j}\left(\overline{g}_j(s,r) - \overline{g}_{iN}(s,r)\right) ds\right)\Bigg\|$$

$$\leq \sum_{i=1}^{m} \left\| \left(\underline{g}_i(t,r) - \underline{g}_{iN}(t,r)\right) + \left(\overline{g}_i(t,r) - \overline{g}_{iN}(t,r)\right) \right\|$$
$$+ \sum_{i=1}^{m} \left\| \sum_{j=1}^{m} \lambda_{i,j} \left(\int_a^{c_{i,j}} k_{i,j} \left[\left(\underline{g}_j(s,r) - \underline{g}_{jN}(s,r)\right) \right.\right.\right.$$
$$\left. + \left(\overline{g}_i(s,r) - \overline{g}_{iN}(s,r)\right) \right] ds$$
$$+ \int_{c_{i,j}}^{b} k_{i,j} \left[\left(\underline{g}_j(s,r) - \underline{g}_{jN}(s,r)\right) \right.$$
$$\left.\left.\left. + \left(\overline{g}_i(s,r) - \overline{g}_{iN}(s,r)\right) \right] ds \right) \right\|$$

$$\leq \sum_{i=1}^{m} \left\| \left(\underline{g}_i(t,r) - \underline{g}_{iN}(t,r)\right) + \left(\overline{g}_i(t,r) - \overline{g}_{iN}(t,r)\right) \right\|$$
$$+ \sum_{i=1}^{m} \sum_{j=1}^{m} \left(\left\|\lambda_{i,j}\right\| . \left\| \int_a^{b} k_{i,j} \left[\left(\underline{g}_j(s,r) - \underline{g}_{jN}(s,r)\right) \right.\right.\right.$$
$$\left.\left.\left. + \left(\overline{g}_i(s,r) - \overline{g}_{iN}(s,r)\right) \right] ds \right\| \right)$$

$$\leq \sum_{i=1}^{m} \left\| \left(\underline{g}_i(t,r) - \underline{g}_{iN}(t,r)\right) + \left(\overline{g}_i(t,r) - \overline{g}_{iN}(t,r)\right) \right\|$$
$$+ \sum_{i=1}^{m} \sum_{j=1}^{m} \left(\left\|\lambda_{i,j}\right\| . \int_a^{b} \left\|k_{i,j}\right\| . \left\| \left(\underline{g}_j(s,r) - \underline{g}_{jN}(s,r)\right) \right.\right.$$
$$\left.\left. + \left(\overline{g}_i(s,r) - \overline{g}_{iN}(s,r)\right) \right\| ds \right)$$

$$\leq \sum_{i=1}^{m} \left\| \left(\underline{g}_i(t,r) - \underline{g}_{iN}(t,r)\right) + \left(\overline{g}_i(t,r) - \overline{g}_{iN}(t,r)\right) \right\|$$
$$+ \sum_{i=1}^{m} \sum_{j=1}^{m} \left(\left\|\lambda_{i,j}\right\| . \int_a^{b} \left\|k_{i,j}\right\| . \left[\left\| \underline{g}_j(s,r) - \underline{g}_{jN}(s,r) \right\| \right.\right.$$
$$\left.\left. + \left\| \overline{g}_i(s,r) - \overline{g}_{iN}(s,r) \right\| \right] ds \right).$$

We know that the kernel $k_{i,j}(s,t)$ is a continuous function on $[a,b]$ so $\|k_{i,j}\|$ is bounded, $\left\|\underline{g}_j(s,r) - \underline{g}_{jN}(s,r)\right\| \to 0$ and

$\|\overline{g}_i(s,r) - \overline{g}_{iN}(s,r)\| \to 0$ as $N \to \infty, i.e. \|e_N\| \to 0.$

That means that if N is large enough then the error function $e_N(t,r)$ becomes zero.

4.2 Trapezoidal Method

We compute the Riemann Integral in definition (1.13) of $\underline{g}(t,r)$ and $\overline{g}(t,r)$ by applying the trapezoidal rule, so we consider $\underline{g}(t,r)$ and $\overline{g}(t,r)$ over the interval $[a,b]$, then suppose the interval $[a,b]$ is subdivided into n subintervals of equal width $h = \frac{b-a}{n}$ by using equally spaced nodes:

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\,,$$

$$t_i = a + ih \;, t_i - t_{i-1} = h\,, 1 \leq i \leq n \tag{4.12}$$

We define [31]

$$\underline{S}_n(r) = h\left[\frac{\underline{f}(a,r) + \underline{f}(b,r)}{2} + \sum_{i=1}^{n-1} \underline{f}(t_i,r)\right]$$

$$\overline{S}_n(r) = h\left[\frac{\overline{f}(a,r) + \overline{f}(b,r)}{2} + \sum_{i=1}^{n-1} \overline{f}(t_i,r)\right] \tag{4.13}$$

Then, for arbitrary fixed r, we have

$$\lim_{n\to\infty} \underline{S}_n(r) = \underline{g}(r) = \int_a^b \underline{f}(t,r)\,dt$$

$$\lim_{n\to\infty} \overline{S}_n(r) = \overline{g}(r) = \int_a^b \overline{f}(t,r)\,dt \tag{4.14}$$

Theorem (4.2) [32]: If $f(t)$ is continuous in the metric D, then $\underline{S}_n(r)$ and $\overline{S}_n(r)$ converge uniformly in r to $\underline{g}(t,r)$ and $\overline{g}(t,r)$ respectively.

Proof: The definite integral of $g(t)$ guaranteed its existence by the continuity of $g(t)$. If

$$\lim_{n\to\infty} [\max_{1\le i\le n} (t_i - t_{i-1})] = 0 \tag{4.15}$$

then R_p of definition (1.13) converges to the definite integral of $g(t)$.

Now, for arbitrary $R_p(r) = \left(\underline{R}_p(r), \overline{R}_p(r)\right)$ and $g(r) = \left(\underline{g}(r), \overline{g}(r)\right)$, we have

$$D(R_p, g) = \sup_{0\le r\le 1} \left\{\max\left[\left|\underline{R}_p(r) - \underline{g}(r)\right|, \left|\overline{R}_p(r) - \overline{g}(r)\right|\right]\right\} \tag{4.16}$$

and since

$$\lim_{n\to\infty} D(R_p, g) = 0,\ \max_{1\le i\le n} (t_i - t_{i-1}) \to 0 \tag{4.17}$$

we obtain that R_p converges uniformly to g. Knowing that S_n is particular case of R_p, consequently $\underline{S}_n(r)$ and $\overline{S}_n(r)$ converges uniformly to $\underline{g}(r)$ and $\overline{g}(r)$ respectively.

Definition (4.1) [31]: A fuzzy number $u(r) = \left(\underline{u}(r), \overline{u}(r)\right)$ belongs to CE is defined as

$$CE = \left\{\left(\underline{u}(r), \overline{u}(r)\right) : \underline{u}(r), \overline{u}(r) \in C[0,1]\right\}$$

where CE is a subclass of E. There is a necessary and sufficient condition for a fuzzy number $u(r)$ to belong to CE is given in [28].

Theorem (4.3) [32]: Let $g(t,r)=\left(\underline{g}(t,r),\overline{g}(t,r)\right)$ be a fuzzy continuous function in t for fixed r and belong to CE, then its approximate solutions $\underline{S}_n(r)$ and $\overline{S}_n(r)$ converges uniformly.

The exact iterative process for finding the exact solution for equation (1.7) is given by

$$g_0(t)=f(t)$$

$$g_m(t)=f(t)+\lambda\int_a^b k(s,t)g_{m-1}(s)\,ds \quad ,m\geq 1 \tag{4.18}$$

However, the numerical process provide us approximate fuzzy function for $g_m(t)$, denote it $S_n^{(m)}$ at the mth iteration using n integration nodes. We have

$$S_n^{(m)}(t,r)=f(t,r)+\lambda\int_a^b k(s,t)S_{n-1}^{(m-1)}\,ds+\delta_n(t,r) \tag{4.19}$$

where $\delta_n(t,r)=\left(\delta_n^{(1)}(t,r),\delta_n^{(2)}(t,r)\right)$ all components uniformly approach 0 as $n,m\to\infty$.

Now, let $\delta_n(t,r)=\left(\underline{\delta}_n(t,r),\overline{\delta}_n(t,r)\right)$ and we neglect $\delta_n(t,r)$ in equation (4.19), then we have

$$\underline{S}_n^{(0)}(t,r)=\underline{f}(t,r)$$

$$\underline{S}_n^{(m)}(t,r)=\underline{f}(t,r)+\lambda\int_a^b k(s,t)\underline{S}_{n-1}^{(m-1)}\,ds$$

and

$$\overline{S}_n^{(0)}(t,r) = \overline{f}(t,r)$$

$$\overline{S}_n^{(m)}(t,r) = \overline{f}(t,r) + \lambda \int_a^b k(s,t)\overline{S}_{n-1}^{(m-1)}\, ds \tag{4.20}$$

Theorem (4.4) [32]: Let $S_n^{(m)}(t)$ be an approximation to $g_m(t)$ using the trapezoidal rule with m equally spaced integration nodes, then $S_n^{(m)}(t)$ converges uniformly to the unique solution $g(t)$ when $n, m \to \infty$.

Proof: see [31].

Chapter Five

Numerical Examples and Results

Chapter Five

Numerical Examples and Results

In this chapter we consider some numerical test cases to illustrate the numerical methods presented in chapter four. These include: Taylor expansion method and trapezoidal method. We will use algorithms and MAPLE software for our numerical computations then draw a comparison between approximate solutions and the exact ones.

Numerical example (5.1): (Taylor expansion method) The following fuzzy Fredholm integral equations:

$$\underline{g}(t,r) = \frac{1}{2}(r+1).t + \int_0^2 \frac{1}{4} t.\underline{g}(s,r)ds$$

$$\overline{g}(t,r) = \frac{1}{2}(3-r).t + \int_0^2 \frac{1}{4} t.\overline{g}(s,r)\,ds \qquad (5.1)$$

have the exact solutions

$$\underline{g}(t,r) = (r+1).t$$

$$\overline{g}(t,r) = (3-r).t \qquad (5.2)$$

Here we expand the unknown functions in Taylor series at $z = \frac{1}{2}$.

The following algorithm implements the Taylor expansion method using MAPLE software.

Algorithm (5.1)

1. input $a, b, \lambda_{i,j}, z, m, k_{i,j}(s,t), \underline{f}_i(t,r), \overline{f}_i(t,r)$

2. input the Taylor expansion degree N

3. calculate $\frac{\partial^{(p)} k_{i,j}(s,t)}{\partial t^p}, \frac{\partial^{(p)} \underline{f}_i(t,r)}{\partial t^p}, \frac{\partial^{(p)} \overline{f}_i(t,r)}{\partial t^p}$ $, p, q = 0, \dots, N$

4. calculate $w_{p,q}^{(i,j)} = \frac{\lambda_{i,j}}{q!} \int_a^{c_{i,j}} \frac{\partial^{(p)} k_{i,j}(s,t)}{\partial t^p} . (s-z)^p ds$ $, i, j = 1 \dots m$

5. calculate $w_{p,q}^{'(i,j)} = \frac{\lambda_{i,j}}{q!} \int_{c_{i,j}}^{b} \frac{\partial^{(p)} k_{i,j}(s,t)}{\partial t^p} . (s-z)^p ds$

6. put $W_{1,1}^{(i,j)} = W_{2,2}^{(i,j)} =$

$$\begin{bmatrix} w_{0,0}^{(i,j)} - 1 & w_{0,1}^{(i,j)} & \dots & w_{0,N-1}^{(i,j)} & w_{0,N}^{(i,j)} \\ w_{1,0}^{(i,j)} & w_{1,1}^{(i,j)} - 1 & \dots & w_{1,N-1}^{(i,j)} & w_{1,N}^{(i,j)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{N-1,0}^{(i,j)} & w_{N-1,1}^{(i,j)} & \dots & w_{N-1,N-1}^{(i,j)} - 1 & w_{N-1,N}^{(i,j)} \\ w_{N,0}^{(i,j)} & w_{N,1}^{(i,j)} & \dots & w_{N,N-1}^{(i,j)} & w_{N,N}^{(i,j)} - 1 \end{bmatrix}$$

7. put

$$W_{1,2}^{'(i,j)} = W_{2,1}^{'(i,j)} = \begin{bmatrix} w_{0,0}^{'(i,j)} & w_{0,1}^{'(i,j)} & \dots & w_{0,N-1}^{'(i,j)} & w_{0,N}^{'(i,j)} \\ w_{1,0}^{'(i,j)} & w_{1,1}^{'(i,j)} & \dots & w_{1,N-1}^{'(i,j)} & w_{1,N}^{'(i,j)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{N-1,0}^{'(i,j)} & w_{N-1,1}^{'(i,j)} & \dots & w_{N-1,N-1}^{'(i,j)} & w_{N-1,N}^{'(i,j)} \\ w_{N,0}^{'(i,j)} & w_{N,1}^{'(i,j)} & \dots & w_{N,N-1}^{'(i,j)} & w_{N,N}^{'(i,j)} \end{bmatrix}$$

8. denote

$$W^{(i,j)} = \begin{bmatrix} W_{1,1}^{(i,j)} & W_{1,2}^{(i,j)} \\ W_{2,1}^{(i,j)} & W_{2,2}^{(i,j)} \end{bmatrix}$$

9. put $F = \begin{bmatrix} -\underline{f}_1(z,r) \\ \vdots \\ -\frac{\partial^{(N)}\underline{f}_i(z,r)}{\partial t^N} \\ -\overline{f}_1(z,r) \\ \vdots \\ -\frac{\partial^{(N)}\overline{f}_m(z,r)}{\partial t^N} \\ \vdots \\ -\underline{f}_m(z,r) \\ \vdots \\ -\frac{\partial^{(N)}\underline{f}_m(z,r)}{\partial t^N} \\ -\overline{f}_m(z,r) \\ \vdots \\ -\frac{\partial^{(N)}\overline{f}_m(z,r)}{\partial t^N} \end{bmatrix}$

10. solve the following linear system

$$WG = F$$

11. Estimate $\underline{g}(z,r), \overline{g}(z,r)$ by computing Taylor expansion for G

$$\underline{g}(t,r) = \sum_{p=0}^{N} \frac{1}{p!} \frac{\partial^{(p)}\underline{g}(z,r)}{\partial t^p}\bigg|_{t=z} (t-z)^p$$

$$\overline{g}(t,r) = \sum_{p=0}^{N} \frac{1}{p!} \frac{\partial^{(p)}\overline{g}(z,r)}{\partial t^p}\bigg|_{t=z} (t-z)^p$$

So we obtain the following results:

$$W=\begin{bmatrix}-\frac{3}{4} & \frac{1}{8} & 0 & 0\\ \frac{1}{2} & -\frac{3}{4} & 0 & 0\\ 0 & 0 & -\frac{3}{4} & \frac{1}{8}\\ 0 & 0 & \frac{1}{2} & -\frac{3}{4}\end{bmatrix}$$

$$F=\begin{bmatrix}-\underline{f}(t,r)\\ -\underline{f}'(t,r)\\ -\overline{f}(t,r)\\ -\overline{f}'(t,r)\end{bmatrix}=\begin{bmatrix}-\frac{1}{2}(r+1).t\\ -\frac{1}{2}(r+1)\\ -\frac{1}{2}(3-r).t\\ -\frac{1}{2}(3-r)\end{bmatrix}_{t=\frac{1}{2}}=\begin{bmatrix}-\frac{1}{4}(r+1)\\ -\frac{1}{2}(r+1)\\ -\frac{1}{4}(3-r)\\ -\frac{1}{2}(3-r)\end{bmatrix}$$

Solving the following linear system

$$WG=F$$

$$\begin{bmatrix}-\frac{3}{4} & \frac{1}{8} & 0 & 0\\ \frac{1}{2} & -\frac{3}{4} & 0 & 0\\ 0 & 0 & -\frac{3}{4} & \frac{1}{8}\\ 0 & 0 & \frac{1}{2} & -\frac{3}{4}\end{bmatrix}G=\begin{bmatrix}-\frac{1}{4}(r+1)\\ -\frac{1}{2}(r+1)\\ -\frac{1}{4}(3-r)\\ -\frac{1}{2}(3-r)\end{bmatrix}$$

we obtain

$$G=\begin{bmatrix}\underline{g}(\frac{1}{2},r)\\ \underline{g}'(\frac{1}{2},r)\\ \overline{g}(\frac{1}{2},r)\\ \overline{g}'(\frac{1}{2},r)\end{bmatrix}=\begin{bmatrix}\frac{1}{2}(r+1)\\ (r+1)\\ \frac{1}{2}(3-r)\\ (3-r)\end{bmatrix}$$

Now,

$$\underline{g}(t,r) = \sum_{p=0}^{N} \frac{1}{p!} \frac{\partial^{(p)} \underline{g}(t,r)}{\partial t^p}\bigg|_{t=\frac{1}{2}} (t-\frac{1}{2})^p = \frac{1}{2}(r+1) + (r+1)\left(t-\frac{1}{2}\right)$$

$$= (r+1).t$$

$$\overline{g}(t,r) = \sum_{p=0}^{N} \frac{1}{p!} \frac{\partial^{(p)} \overline{g}(t,r)}{\partial t^p}\bigg|_{t=\frac{1}{2}} (t-\frac{1}{2})^p = \frac{1}{2}(3-r) + (3-r)\left(t-\frac{1}{2}\right)$$

$$= (3-r).t$$

This agrees with the exact solution (5.2) for equation (5.1). Moreover, for a fixed t we compare the exact and the approximate solutions of equation (5.1) as shown in figure (5.1).

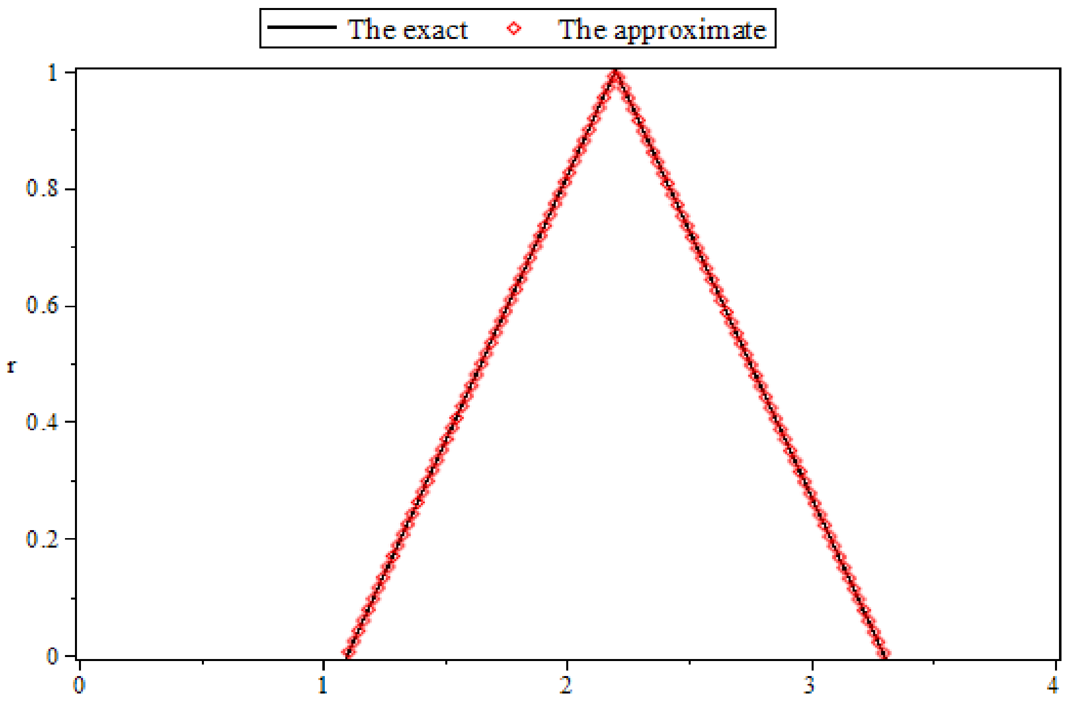

Figure (5.1) the exact and the approximate solution at t=1

Table (5.1) compares the results with the exact solution using Definition (1.11).

Table (5.1): The error resulted by algorithm (5.1) at $t = 1$

r	$\underline{g}_{exact}$	$\overline{g}_{exact}$	$\underline{g}_{approximate}$	$\overline{g}_{approximate}$
0.0	1	3	1	3
0.1	1.1	2.9	1.1	2.9
0.2	1.2	2.8	1.2	2.8
0.3	1.3	2.7	1.3	2.7
0.4	1.4	2.6	1.4	2.6
0.5	1.5	2.5	1.5	2.5
0.6	1.6	2.4	1.6	2.4
0.7	1.7	2.3	1.7	2.3
0.8	1.8	2.2	1.8	2.2
0.9	1.9	2.1	1.9	2.1
1.0	2	2	2	2

These results reveal the accuracy and the great potential of Taylor expansion method for solving equation (5.1) since the

$$error = D\big(g_{exact}(1,r), g_{approx}(1,r)\big) =$$

$$\sup_{0\le r\le 1}\left\{\max\left|\underline{g}_{exact} - \underline{g}_{approx}\right|, \max\left|\overline{g}_{exact} - \overline{g}_{approx}\right|\right\} = 0.$$

Numerical example (5.2): (Taylor expansion method) The following fuzzy Fredholm integral equations:

$$\underline{g}(t,r) = (r+1).(e^{-t} + t - \sin t) + \int_0^1 \frac{1}{2}.e^{s}.\sin t\,.\underline{g}(s,r)ds$$

$$\overline{g}(t,r) = (3-r).(e^{-t} + t - \sin t) + \int_0^1 \frac{1}{2}.e^{s}.\sin t\,.\overline{g}(s,r)\,ds \qquad (5.3)$$

have the exact solutions

$$\underline{g}(t,r) = (r+1).(e^{-t} + t)$$

$$\overline{g}(t,r) = (3-r).(e^{-t} + t) \qquad (5.4)$$

Here we expand the unknown functions in Taylor series at $z = \frac{1}{2}$.

Algorithm (5.1) implements the Taylor expansion method using MAPLE software so we obtain the following results:

$$W_{1,1}^{1,1} = W_{2,2}^{1,1}$$

$$= \begin{bmatrix} -0.588105905 & 0.033765722 & 0.017721040 & 0.000848119 \\ 0.753967085 & -0.938192261 & 0.032438147 & 0.001552471 \\ -0.411894096 & -0.033765722 & -1.017721040 & -0.000848119 \\ -0.753967085 & -0.061807739 & -0.032438147 & -1.001552471 \end{bmatrix}$$

$$W_{1,2}^{1,1} = W_{2,1}^{1,1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

hence,

$$W = \begin{bmatrix} W_{1,1}^{1,1} & W_{1,2}^{1,1} \\ W_{2,1}^{1,1} & W_{2,2}^{1,1} \end{bmatrix}$$

$$F=\begin{bmatrix}-\underline{f}(t,r)\\-\underline{f}'(t,r)\\-\underline{f}''(t,r)\\-\underline{f}'''(t,r)\\-\overline{f}(t,r)\\-\overline{f}'(t,r)\\-\overline{f}''(t,r)\\-\overline{f}'''(t,r)\end{bmatrix}=\begin{bmatrix}-(r+1).(e^{-t}+t-\sin t)\\-(r+1).(-e^{-t}+1-\cos t)\\-(r+1).(e^{-t}+\sin t)\\-(r+1).(-e^{-t}+\cos t)\\-(3-r).(e^{-t}+t-\sin t)\\-(3-r).(-e^{-t}+t-\cos t)\\-(3-r).(e^{-t}+\sin t)\\-(3-r).(-e^{-t}+\cos t)\end{bmatrix}_{t=\frac{1}{2}}$$

$$F=\begin{bmatrix}-0.627105121\,(r+1)\\0.484113225\,(r+1)\\-1.085956198\,(r+1)\\-0.271051902\,(r+1)\\-0.627105121\,(3-r)\\0.484113222\,(3-r)\\-1.085956198\,(3-r)\\-0.271051902\,(3-r)\end{bmatrix}$$

Solving the following linear system

$$WG=F$$

we obtain

$$G=\begin{bmatrix}\underline{g}(0.5,r)\\\underline{g}'(0.5,r)\\\underline{g}''(0.5,r)\\\underline{g}'''(0.5,r)\\\overline{g}(0.50,r)\\\overline{g}'(0.5,r)\\\overline{g}''(0.5,r)\\\overline{g}'''(0.5,r)\end{bmatrix}=\begin{bmatrix}1.106288029\,(r+1)\\0.393025207\,(r+1)\\0.606773291\,(r+1)\\0.606086526\,(r+1)\\1.106288029\,(3-r)\\0.393025207\,(3-r)\\0.606773291\,(3-r)\\-0.606086526\,(3-r)\end{bmatrix}$$

Now,

$$\underline{g}(t,r) = \sum_{p=0}^{N} \frac{1}{p!} \frac{\partial^{(p)} \underline{g}(t,r)}{\partial t^{p}} \bigg|_{t=\frac{1}{2}} (t - \frac{1}{2})^{p}$$

$$= 0.998248889\,(r+1) + 0.013877746\,(r+1).t$$

$$+ 0.454908277\,(r+1).t^{2} - 0.101014421\,(r+1).t^{3}$$

$$\overline{g}(t,r) = \sum_{p=0}^{N} \frac{1}{p!} \frac{\partial^{(p)} \overline{g}(t,r)}{\partial t^{p}} \bigg|_{t=\frac{1}{2}} (t - \frac{1}{2})^{p}$$

$$= 0.998248889\,(3-r) + 0.013877746\,(3-r).t$$

$$+ 0.454908277\,(3-r).t^{2} - 0.101014421\,(3-r).t^{3}$$

Figure (5.2) compares the exact and approximate solutions for a fixed $t = 1$.

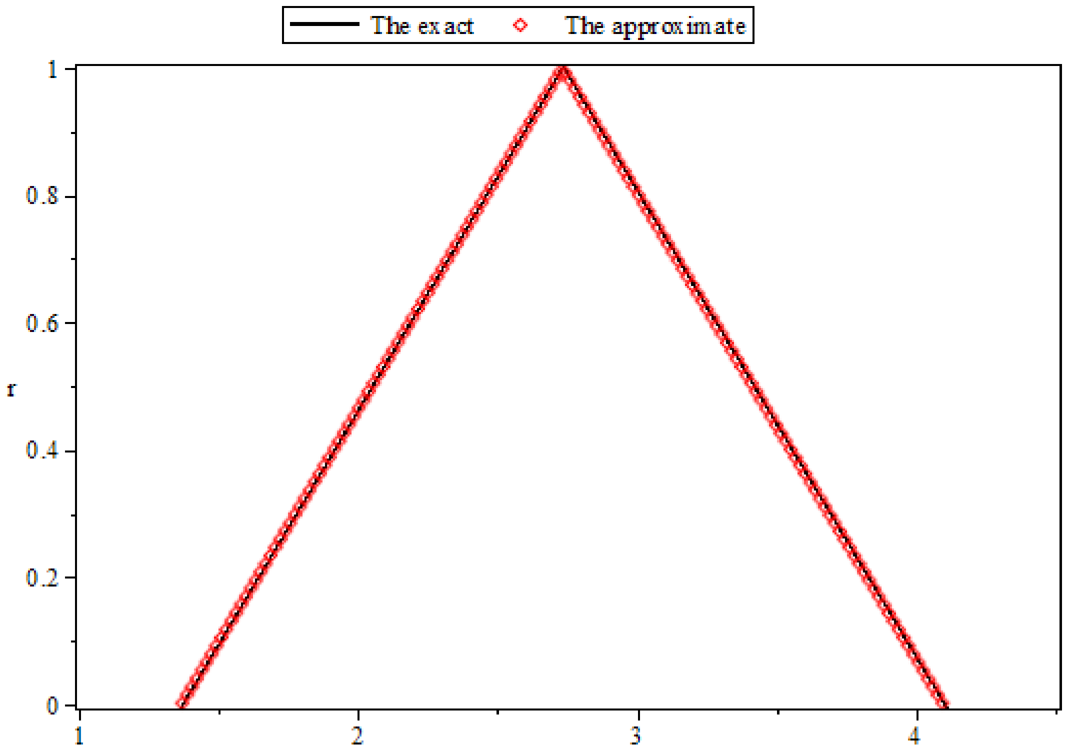

Figure (5.2) the exact and the approximate solution at t=1

Table (5.2) uses Definition (1.11) to compare the approximate solution with exact solution (5.4).

Table (5.2): The error resulted by algorithm (5.1) at $t = 1$.

r	$\underline{g}_{exact}$	$\overline{g}_{exact}$	$\underline{g}_{approximate}$	$\overline{g}_{approximate}$	$error = D(g_{exact}, g_{approx})$
0.0	1.367879441	4.103638324	1.366020491	4.098061473	$5.576851372 \times 10^{-3}$
0.1	1.504667385	3.966850379	1.502622540	3.961459423	$5.390955481 \times 10^{-3}$
0.2	1.641455329	3.830062435	1.639224589	3.824857374	$5.205060586 \times 10^{-3}$
0.3	1.778243274	3.693274491	1.775826638	3.688255325	$5.019165700 \times 10^{-3}$
0.4	1.915031218	3.556486547	1.912428687	3.551653276	$4.833270801 \times 10^{-3}$
0.5	2.051819162	3.419698603	2.049030737	3.415051227	$4.647375913 \times 10^{-3}$
0.6	2.188607106	3.282910659	2.185632786	3.278449178	$4.461481016 \times 10^{-3}$
0.7	2.325395050	3.146122715	2.322234835	3.141847129	$4.275586125 \times 10^{-3}$
0.8	2.462182994	3.009334771	2.458836884	3.005245080	$4.089691232 \times 10^{-3}$
0.9	2.598970938	2.872546826	2.595438933	2.868643031	$3.903795339 \times 10^{-3}$
1.0	2.735758882	2.735758882	2.732040982	2.732040982	$3.717900446 \times 10^{-3}$

These results reveal the accuracy of Taylor expansion method to solve equation (5.3) since the max $error = 5.576851372 \times 10^{-3}$.

Numerical example (5.3): (Trapezoidal method) The fuzzy Fredholm integral equations (5.1) have the exact solution (5.2) where

$n = 51$, on the interval $[0,2]$, $h = \frac{b-a}{n} = 0.03921568627$,

$0 = t_0 \leq t_1 \leq \cdots \leq t_{51} = 2, t_i = i.h,$

The approximate fuzzy function calculated at the 24-th iteration with $n = 51$, the following algorithm implements the trapezoidal rule using MAPLE software.

Algorithm (5.2)

1. input $a, b, \lambda\ , k(s,t), \underline{f}(t,r), \overline{f}(t,r), n$, m

2. $h = \frac{b-a}{n}$

3. $t_0 = a, t_n = b$

4. For $i = 1\ to\ n$, compute $t_i = a + i.h$

5. Compute $\underline{S}_n^{(0)}(t,r) = \underline{f}(t,r)$

$$\underline{S}_{n-1}^{(m-1)}(r) = h\left[\frac{\underline{f}(a,r) + \underline{f}(b,r)}{2} + \sum_{i=1}^{n-2} \underline{f}(t_i,r)\right]$$

$$\underline{S}_n^{(m)}(t,r) = \underline{f}(t,r) + \lambda \int_a^b k(s,t)\underline{S}_{n-1}^{(m-1)}\, ds$$

6. Compute $\overline{S}_n^{(0)}(t,r) = \overline{f}(t,r)$

$$\overline{S}_{n-1}^{(m-1)}(r) = h\left[\frac{\overline{f}(a,r) + \overline{f}(b,r)}{2} + \sum_{i=1}^{n-2} \overline{f}(t_i,r)\right]$$

$$\overline{S}_n^{(m)}(t,r) = \overline{f}(t,r) + \lambda \int_a^b k(s,t)\overline{S}_{n-1}^{(m-1)}\, ds$$

So we obtain the following results:

$$\underline{S}_{51}^{(0)}(t,r) = 0.5.(r+1).t$$

$$\underline{S}_{51}^{(1)}(t,r) = 0.5.(r+1).t + \int_0^2 k(s,t).\underline{S}_{50}^{(0)}(s,r)ds$$

$$= 0.7452902731(r+1).t$$

$$\underline{S}_{51}^{(2)}(t,r) = 0.5.(r+1).t + \int_{0}^{2} k(s,t).\underline{S}_{50}^{(1)}(s,r)ds$$

$$= 0.8656249092\ (r+1).t$$

$$\underline{S}_{51}^{(3)}(t,r) = 0.5.(r+1).t + \int_{0}^{2} k(s,t).\underline{S}_{50}^{(2)}(s,r)ds$$

$$= 0.9246587402\ (r+1).t$$

$$\underline{S}_{51}^{(4)}(t,r) = 0.5.(r+1).t + \int_{0}^{2} k(s,t).\underline{S}_{50}^{(3)}(s,r)ds$$

$$= 0.9536195895\ (r+1).t$$

$$\underline{S}_{51}^{(5)}(t,r) = 0.5.(r+1).t + \int_{0}^{2} k(s,t).\underline{S}_{50}^{(4)}(s,r)ds$$

$$= 0.9678272190\ (r+1).t$$

$$\underline{S}_{51}^{(6)}(t,r) = 0.5.(r+1).t + \int_{0}^{2} k(s,t).\underline{S}_{50}^{(5)}(s,r)ds$$

$$= 0.9747972055\ (r+1).t$$

$$\underline{S}_{51}^{(7)}(t,r) = 0.5.(r+1).t + \int_{0}^{2} k(s,t).\underline{S}_{50}^{(6)}(s,r)ds$$

$$= 0.9782165450\ (r+1).t$$

$$\underline{S}_{51}^{(8)}(t,r) = 0.5.(r+1).t + \int_{0}^{2} k(s,t).\underline{S}_{50}^{(7)}(s,r)ds$$

$$= 0.9798940068\ (r+1).t$$

$$\underline{S}_{51}^{(9)}(t,r) = 0.5.(r+1).t + \int_0^2 k(s,t).\underline{S}_{50}^{(8)}(s,r)ds$$

$$= 0.9807169370\ (r+1).t$$

$$\underline{S}_{51}^{(10)}(t,r) = 0.5.(r+1).t + \int_0^2 k(s,t).\underline{S}_{50}^{(9)}(s,r)ds$$

$$= 0.9811206502\ (r+1).t$$

$$\underline{S}_{51}^{(11)}(t,r) = 0.5.(r+1).t + \int_0^2 k(s,t).\underline{S}_{50}^{(10)}(s,r)ds$$

$$= 0.9813187035\ (r+1).t$$

$$\underline{S}_{51}^{(12)}(t,r) = 0.5.(r+1).t + \int_0^2 k(s,t).\underline{S}_{50}^{(11)}(s,r)ds$$

$$= 0.9814158655\ (r+1).t$$

$$\underline{S}_{51}^{(13)}(t,r) = 0.5.(r+1).t + \int_0^2 k(s,t).\underline{S}_{50}^{(12)}(s,r)ds$$

$$= 0.9814635310\ (r+1).t$$

$$\underline{S}_{51}^{(14)}(t,r) = 0.5.(r+1).t + \int_0^2 k(s,t).\underline{S}_{50}^{(13)}(s,r)ds$$

$$= 0.9814869148\ (r+1).t$$

$$\underline{S}_{51}^{(15)}(t,r) = 0.5.(r+1).t + \int_{0}^{2} k(s,t).\underline{S}_{50}^{(14)}(s,r)ds$$

$$= 0.9814983865\,(r+1).t$$

$$\underline{S}_{51}^{(16)}(t,r) = 0.5.(r+1).t + \int_{0}^{2} k(s,t).\underline{S}_{50}^{(15)}(s,r)ds$$

$$= 0.9815040140\,(r+1).t$$

$$\underline{S}_{51}^{(17)}(t,r) = 0.5.(r+1).t + \int_{0}^{2} k(s,t).\underline{S}_{50}^{(16)}(s,r)ds$$

$$= 0.9815067750\,(r+1).t$$

$$\underline{S}_{51}^{(18)}(t,r) = 0.5.(r+1).t + \int_{0}^{2} k(s,t).\underline{S}_{50}^{(17)}(s,r)ds$$

$$= 0.9815081295\,(r+1).t$$

$$\underline{S}_{51}^{(19)}(t,r) = 0.5.(r+1).t + \int_{0}^{2} k(s,t).\underline{S}_{50}^{(18)}(s,r)ds$$

$$= 0.9815087942\,(r+1).t$$

$$\underline{S}_{51}^{(20)}(t,r) = 0.5.(r+1).t + \int_{0}^{2} k(s,t).\underline{S}_{50}^{(19)}(s,r)ds$$

$$= 0.9815091202\,(r+1).t$$

$$\underline{S}_{51}^{(21)}(t,r) = 0.5.(r+1).t + \int_{0}^{2} k(s,t).\underline{S}_{50}^{(20)}(s,r)ds$$

$$= 0.9815092802\ (r+1).t$$

$$\underline{S}_{51}^{(22)}(t,r) = 0.5.(r+1).t + \int_0^2 k(s,t).\underline{S}_{50}^{(21)}(s,r)ds$$

$$= 0.9815093585\ (r+1).t$$

$$\underline{S}_{51}^{(23)}(t,r) = 0.5.(r+1).t + \int_0^2 k(s,t).\underline{S}_{50}^{(22)}(s,r)ds$$

$$= 0.9815093968\ (r+1).t$$

$$\underline{S}_{51}^{(24)}(t,r) = 0.5.(r+1).t + \int_0^2 k(s,t).\underline{S}_{50}^{(23)}(s,r)ds$$

$$= 0.9851094158\ (r+1).t$$

and

$$\overline{S}_{51}^{(0)}(t,r) = 0.5.(3-r).t$$

$$\overline{S}_{51}^{(1)}(t,r) = 0.5.(3-r).t + \int_0^2 k(s,t).\overline{S}_{50}^{(0)}(s,r)ds$$

$$= 0.7452902731\ (3-r).t$$

$$\overline{S}_{51}^{(2)}(t,r) = 0.5.(3-r).t + \int_0^2 k(s,t).\overline{S}_{50}^{(1)}(s,r)ds$$

$$= 0.8656249092\ (3-r).t$$

$$\overline{S}_{51}^{(3)}(t,r) = 0.5.(3-r).t + \int_0^2 k(s,t).\overline{S}_{50}^{(2)}(s,r)ds$$

$$= 0.9246587402\ (3.000000002 - r).t$$

$$\overline{S}_{51}^{(4)}(t,r) = 0.5.(3-r).t + \int_0^2 k(s,t).\overline{S}_{50}^{(3)}(s,r)ds$$

$$= 0.9536195895\ (3.000000002 - r).t$$

$$\overline{S}_{51}^{(5)}(t,r) = 0.5.(3-r).t + \int_0^2 k(s,t).\overline{S}_{50}^{(4)}(s,r)ds$$

$$= 0.9678272190\ (3 - r).t$$

$$\overline{S}_{51}^{(6)}(t,r) = 0.5.(3-r).t + \int_0^2 k(s,t).\overline{S}_{50}^{(5)}(s,r)ds$$

$$= 0.9747972055\ (2.999999999 - r).t$$

$$\overline{S}_{51}^{(7)}(t,r) = 0.5.(3-r).t + \int_0^2 k(s,t).\overline{S}_{50}^{(6)}(s,r)ds$$

$$= 0.9782165450\ (3.000000001 - r).t$$

$$\overline{S}_{51}^{(8)}(t,r) = 0.5.(3-r).t + \int_0^2 k(s,t).\overline{S}_{50}^{(7)}(s,r)ds$$

$$= 0.9798940068\ (3 - r).t$$

$$\overline{S}_{51}^{(9)}(t,r) = 0.5.(3-r).t + \int_0^2 k(s,t).\overline{S}_{50}^{(8)}(s,r)ds$$

$$= 0.9807169370\ (2.999999999 - r).t$$

$$\overline{S}_{51}^{(10)}(t,r) = 0.5.(3-r).t + \int_0^2 k(s,t).\overline{S}_{50}^{(9)}(s,r)ds$$

$$= 0.9811206502\,(3.000000001 - r).t$$

$$\overline{S}_{51}^{(11)}(t,r) = 0.5.(3-r).t + \int_0^2 k(s,t).\overline{S}_{50}^{(10)}(s,r)ds$$

$$= 0.9813187035\,(3.000000003 - r).t$$

$$\overline{S}_{51}^{(12)}(t,r) = 0.5.(3-r).t + \int_0^2 k(s,t).\overline{S}_{50}^{(11)}(s,r)ds$$

$$= 0.9814158655\,(2.999999999 - r).t$$

$$\overline{S}_{51}^{(13)}(t,r) = 0.5.(3-r).t + \int_0^2 k(s,t).\overline{S}_{50}^{(12)}(s,r)ds$$

$$= 0.9814635310\,(3.000000001 - r).t$$

$$\overline{S}_{51}^{(14)}(t,r) = 0.5.(3-r).t + \int_0^2 k(s,t).\overline{S}_{50}^{(13)}(s,r)ds$$

$$= 0.9814869148\,(3 - r).t$$

$$\overline{S}_{51}^{(15)}(t,r) = 0.5.(3-r).t + \int_0^2 k(s,t).\overline{S}_{50}^{(14)}(s,r)ds$$

$$= 0.9814983865\,(20999999999 - r).t$$

$$\overline{S}_{51}^{(16)}(t,r) = 0.5.(3-r).t + \int_0^2 k(s,t).\overline{S}_{50}^{(15)}(s,r)ds$$

$$= 0.9815040140\ (3.000000001 - r).t$$

$$\overline{S}_{51}^{(17)}(t,r) = 0.5.(3-r).t + \int_0^2 k(s,t).\overline{S}_{50}^{(16)}(s,r)ds$$

$$= 0.9815067750\ (3.000000001 - r).t$$

$$\overline{S}_{51}^{(18)}(t,r) = 0.5.(3-r).t + \int_0^2 k(s,t).\overline{S}_{50}^{(17)}(s,r)ds$$

$$= 0.9815081295\ (2.999999999 - r).t$$

$$\overline{S}_{51}^{(19)}(t,r) = 0.5.(3-r).t + \int_0^2 k(s,t).\overline{S}_{50}^{(18)}(s,r)ds$$

$$= 0.9815087942\ (2.999999999 - r).t$$

$$\overline{S}_{51}^{(20)}(t,r) = 0.5.(3-r).t + \int_0^2 k(s,t).\overline{S}_{50}^{(19)}(s,r)ds$$

$$= 0.9815091202\ (2.999999999 - r).t$$

$$\overline{S}_{51}^{(21)}(t,r) = 0.5.(3-r).t + \int_0^2 k(s,t).\overline{S}_{50}^{(20)}(s,r)ds$$

$$= 0.9815092802\ (2.999999999 - r).t$$

$$\overline{S}_{51}^{(22)}(t,r) = 0.5.(3-r).t + \int_0^2 k(s,t).\overline{S}_{50}^{(21)}(s,r)ds$$

$$= 0.9815093585\ (2.999999999 - r).t$$

$$\overline{S}_{51}^{(23)}(t,r) = 0.5.(3-r).t + \int_{0}^{2} k(s,t).\overline{S}_{50}^{(22)}(s,r)ds$$

$$= 0.9815093968\,(3.000000001 - r).t$$

$$\overline{S}_{51}^{(24)}(t,r) = 0.5.(3-r).t + \int_{0}^{2} k(s,t).\overline{S}_{50}^{(23)}(s,r)ds$$

$$= 0.9815094158\,(3.000000001 - r).t$$

Figure (5.3) compares both the exact and the approximate solutions for a fixed $t = 1$.

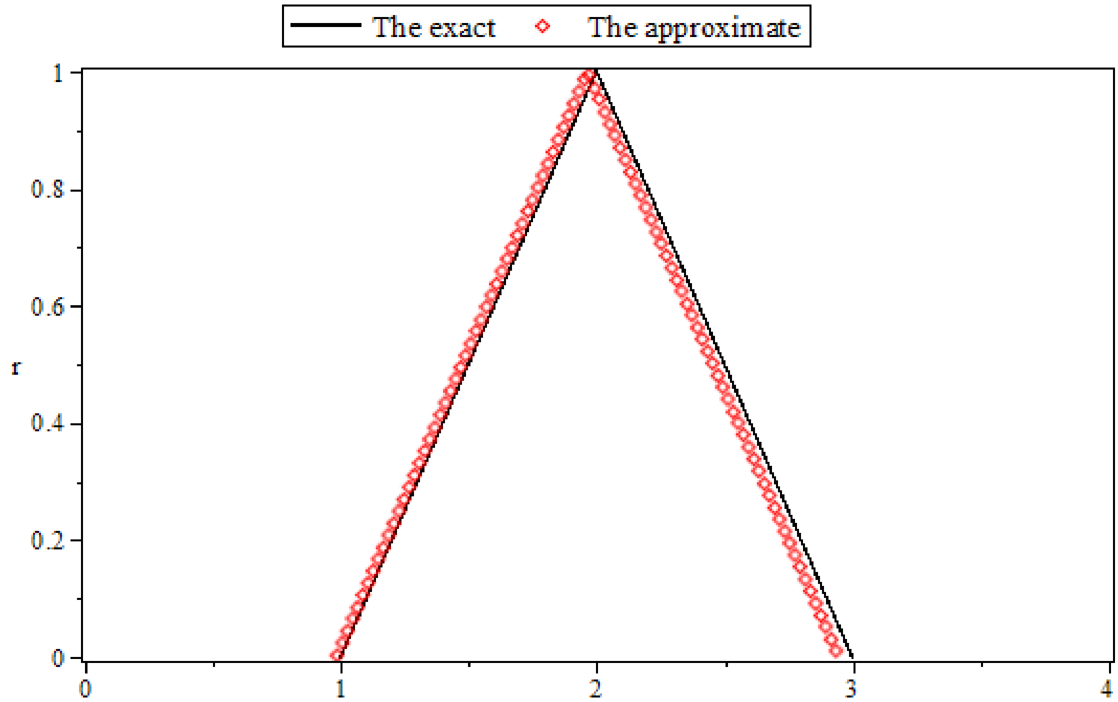

Figure (5.3) the exact and the approximate solution at t=1

We use Definition (1.11) to compare the results of example (5.3) with the exact solution (5.2) as shown in table (5.3) with $n = 51$.

Table (5.3): The error resulted by algorithm (5.2) at $t = 1$.

r	$\underline{g}_{exact}$	$\overline{g}_{exact}$	$\underline{g}_{approximate}$	$\overline{g}_{approximate}$	$error = D(g_{exact}, g_{approx})$
0.0	1	3	$9.815094158 \times 10^{-1}$	2.944528248	$5.547175204 \times 10^{-2}$
0.1	1.1	2.9	1.079660357	2.846377306	$5.362269399 \times 10^{-2}$
0.2	1.2	2.8	1.177811299	2.748226365	$5.177363501 \times 10^{-2}$
0.3	1.3	2.7	1.275962240	2.650075423	$4.992457700 \times 10^{-2}$
0.4	1.4	2.6	1.374113182	2.551924482	$4.807551802 \times 10^{-2}$
0.5	1.5	2.5	1.472264124	2.453773540	$4.622646001 \times 10^{-2}$
0.6	1.6	2.4	1.570415065	2.355622598	$4.437740202 \times 10^{-2}$
0.7	1.7	2.3	1.668566007	2.257471657	$4.252834300 \times 10^{-2}$
0.8	1.8	2.2	1.766716948	2.159320715	$4.067928498 \times 10^{-2}$
0.9	1.9	2.1	1.864867890	2.061169774	$3.883022604 \times 10^{-2}$
1.0	2.0	2.0	1.963018832	1.963018832	$3.698116803 \times 10^{-2}$

These results reveal the effeciency of trapezoidal method to solve equation (5.1) since the max $error = 5.547175204 \times 10^{-2}$.

Numerical example (5.4): (Trapezoidal method) The fuzzy Fredholm integral equations (5.3) have the exact solution (5.4) where

$n = 51$, on the interval $[0,1]$, $h = \frac{b-a}{n} = 0.01960784314$,

$0 = t_0 \leq t_1 \leq \cdots \leq t_{51} = 1, t_i = i.h,$

The approximate fuzzy function calculated at the 24-th iteration with $n = 51$, algorithm (5.2) implements the trapezoidal method using MAPLE software. Then we obtain the following results:

$\underset{\sim}{S}_{51}^{(0)}(t,r) = (r+1).(e^{-t} + t - \sin t)$

$\underset{\sim}{S}_{51}^{(1)}(t,r) = (r+1).[e^{-t} + t] - 0.4311053600\,(r+1)\sin t$

$\underset{\sim}{S}_{51}^{(2)}(t,r) = (r+1).[e^{-t} + t] - 0.2143916095\,(r+1)\sin t$

$\underset{\sim}{S}_{51}^{(3)}(t,r) = (r+1).[e^{-t} + t] - (0.131837044\,r + 0.131837045)\sin t$

$\underset{\sim}{S}_{51}^{(4)}(t,r) = (r+1).[e^{-t} + t] - 0.1003888490\,(r+1)\sin t$

$\underset{\sim}{S}_{51}^{(5)}(t,r) = (r+1).[e^{-t} + t] - (0.088409027\,r + 0.088409028)\sin t$

$\underset{\sim}{S}_{51}^{(6)}(t,r) = (r+1).[e^{-t} + t] - 0.0838454550\,(r+1)\sin t$

$\underset{\sim}{S}_{51}^{(7)}(t,r) = (r+1).[e^{-t} + t] - 0.0821070160\,(r+1)\sin t$

$\underset{\sim}{S}_{51}^{(8)}(t,r) = (r+1).[e^{-t} + t] - 0.0814447775\,(r+1)\sin t$

$\underset{\sim}{S}_{51}^{(9)}(t,r) = (r+1).[e^{-t} + t] - 0.0811923050\,(r+1)\sin t$

$\underset{\sim}{S}_{51}^{(10)}(t,r) = (r+1).[e^{-t} + t] - 0.0810964055\,(r+1)\sin t$

$\underset{\sim}{S}_{51}^{(11)}(t,r) = (r+1).[e^{-t} + t] - 0.0810597975\,(r+1)\sin t$

$\underset{\sim}{S}_{51}^{(12)}(t,r) = (r+1).[e^{-t} + t] - 0.0810458515\,(r+1)\sin t$

$\underset{\sim}{S}_{51}^{(13)}(t,r) = (r+1).[e^{-t} + t] - 0.0810405390\,(r+1)\sin t$

$\underset{\sim}{S}_{51}^{(14)}(t,r) = (r+1).[e^{-t} + t] - 0.0810385145\,(r+1)\sin t$

$\underset{\sim}{S}_{51}^{(15)}(t,r) = (r+1).[e^{-t} + t] - (0.081037744\,r + 0.018037745)\sin t$

$\underset{\sim}{S}_{51}^{(16)}(t,r) = (r+1).[e^{-t} + t] - (0.081037450\,r + 0.081037452)\sin t$

$$\underline{S}_{51}^{(17)}(t,r) = (r+1).[e^{-t}+t] - (0.081037338\,r + 0.081037339)\sin t$$

$$\underline{S}_{51}^{(18)}(t,r) = (r+1).[e^{-t}+t] - (0.081037296\,r + 0.081037297)\sin t$$

$$\underline{S}_{51}^{(19)}(t,r) = (r+1).[e^{-t}+t] - (0.081037280\,r + 0.081037281)\sin t$$

$$\underline{S}_{51}^{(20)}(t,r) = (r+1).[e^{-t}+t] - (0.081037274\,r + 0.081037275)\sin t$$

$$\underline{S}_{51}^{(21)}(t,r) = (r+1).[e^{-t}+t] - 0.0810372715\,(\,r+1)\sin t$$

$$\underline{S}_{51}^{(22)}(t,r) = (r+1).[e^{-t}+t] - 0.0810372705\,(\,r+1)\sin t$$

$$\underline{S}_{51}^{(23)}(t,r) = (r+1).[e^{-t}+t] - 0.0810372705\,(\,r+1)\sin t$$

$$\underline{S}_{51}^{(24)}(t,r) = (r+1).[e^{-t}+t] - 0.0810372705\,(\,r+1)\sin t$$

and

$$\overline{S}_{51}^{(0)}(t,r) = (3-r).(e^{-t}+t-\sin t)$$

$$\overline{S}_{51}^{(1)}(t,r) = (3-r).[e^{-t}+t] - (1.29331608 - 0.431105360\,r)\sin t$$

$$\overline{S}_{51}^{(2)}(t,r) = (3-r).[e^{-t}+t] - (0.64317483 - 0.2114391610\,r)\sin t$$

$$\overline{S}_{51}^{(3)}(t,r) = (3-r).[e^{-t}+t] - (0.39551113 - 0.1318370445\,r)\sin t$$

$$\overline{S}_{51}^{(4)}(t,r) = (3-r).[e^{-t}+t] - (0.30116655 - 0.1003888490\,r)\sin t$$

$$\overline{S}_{51}^{(5)}(t,r) = (3-r).[e^{-t}+t] - (0.265227087 - 0.088409028\,r)\sin t$$

$$\overline{S}_{51}^{(6)}(t,r) = (3-r).[e^{-t}+t] - (0.251536364 - 0.0838454545\,r)\sin t$$

$$\overline{S}_{51}^{(7)}(t,r) = (3-r).[e^{-t}+t] - (0.246321045 - 0.0821070155\,r)\sin t$$

$$\overline{S}_{51}^{(8)}(t,r) = (3-r).[e^{-t}+t] - (0.244334334 - 0.0814447775\,r)\sin t$$

$$\overline{S}_{51}^{(9)}(t,r) = (3-r).[e^{-t}+t] - (0.243577158 - 0.0811925060\,r)\sin t$$

$$\overline{S}_{51}^{(10)}(t,r) = (3-r).[e^{-t}+t] - (0.243289219 - 0.0810964060\,r)\sin t$$

$$\overline{S}_{51}^{(11)}(t,r) = (3-r).[e^{-t}+t] - (0.243179392 - 0.0805979750\,r)\sin t$$

$$\overline{S}_{51}^{(12)}(t,r) = (3-r).[e^{-t}+t] - (0.243137552 - 0.0810458515\,r)\sin t$$

$$\overline{S}_{51}^{(13)}(t,r) = (3-r).[e^{-t}+t] - (0.243121619 - 0.0810405390\,r)\sin t$$

$$\overline{S}_{51}^{(14)}(t,r) = (3-r).[e^{-t}+t] - (0.243115544 - 0.0810385155\,r)\sin t$$

$$\overline{S}_{51}^{(15)}(t,r) = (3-r).[e^{-t}+t] - (0.243113232 - 0.0810377450\,r)\sin t$$

$$\overline{S}_{51}^{(16)}(t,r) = (3-r).[e^{-t}+t] - (0.243112352 - 0.0810374510\,r)\sin t$$

$$\overline{S}_{51}^{(17)}(t,r) = (3-r).[e^{-t}+t] - (0.243112016 - 0.0103733850\,r)\sin t$$

$$\overline{S}_{51}^{(18)}(t,r) = (3-r).[e^{-t}+t] - (0.243111886 - 0.0810372945\,r)\sin t$$

$$\overline{S}_{51}^{(19)}(t,r) = (3-r).[e^{-t}+t] - (0.243111828 - 0.0810372790\,r)\sin t$$

$$\overline{S}_{51}^{(20)}(t,r) = (3-r).[e^{-t}+t] - (0.243111822 - 0.0810372775\,r)\sin t$$

$$\overline{S}_{51}^{(21)}(t,r) = (3-r).[e^{-t}+t] - (0.243111814 - 0.0810372715\,r)\sin t$$

$$\overline{S}_{51}^{(22)}(t,r) = (3-r).[e^{-t}+t] - (0.243111811 - 0.0810372705\,r)\sin t$$

$$\overline{S}_{51}^{(23)}(t,r) = (3-r).[e^{-t}+t] - (0.243111808 - 0.0810372705\,r)\sin t$$

$$\overline{S}_{51}^{(24)}(t,r) = (3-r).[e^{-t}+t] - (0.243111808 - 0.0810372705\,r)\sin t$$

Figure (5.4) compares both the exact and the approximate solutions for a fixed $t = 1$.

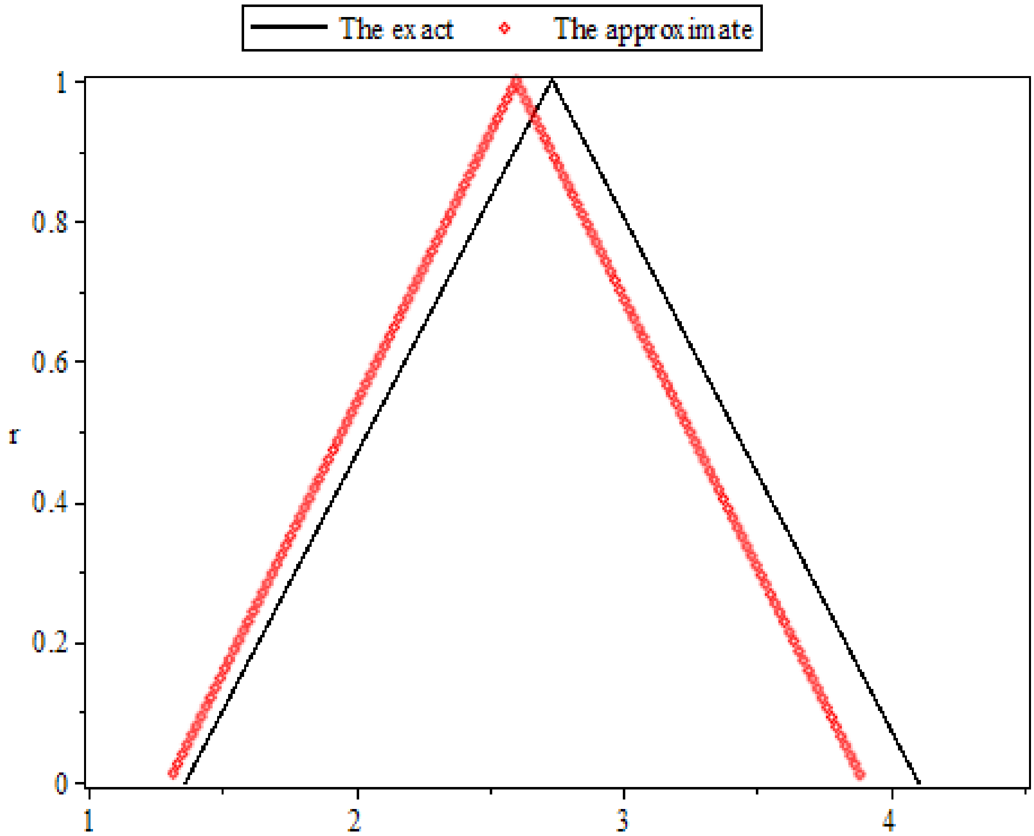

Figure (5.4) the exact and the approximate solution at t=1

Table (5.4) shows a comparison between the results and the exact solution (5.4) using Definition (1.11) with $n = 51$.

Table (5.4): The error resulted by algorithm (5.2) at $t = 1$.

r	$\underline{g}_{exact}$	$\overline{g}_{exact}$	$\underline{g}_{approximate}$	$\overline{g}_{approximate}$	$error = D(g_{exact}, g_{approx})$
0.0	1.367879441	4.103638323	1.299688929	3.899066790	$2.045715329 \times 10^{-1}$
0.1	1.504667385	3.966850379	1.429657822	3.769097898	$1.977524810 \times 10^{-1}$
0.2	1.641455329	3.830062435	1.559626715	3.639129005	$1.909334300 \times 10^{-1}$
0.3	1.778243274	3.693274491	1.689595608	3.509160112	$1.841143791 \times 10^{-1}$
0.4	1.915031218	3.556486547	1.819564500	3.379191219	$1.772953280 \times 10^{-1}$
0.5	2.051819162	3.419698603	1.949533394	3.249222325	$1.704762770 \times 10^{-1}$
0.6	2.188607106	3.282910659	2.079502287	3.119253433	$1.636572250 \times 10^{-1}$
0.7	2.325395050	3.146122715	2.209471180	2.989284540	$1.568381739 \times 10^{-1}$
0.8	2.462182994	3.009334771	2.339440073	2.859315647	$1.500191229 \times 10^{-1}$
0.9	2.598970938	2.872546826	2.469408966	2.729346754	$1.432000720 \times 10^{-1}$
1.0	2.735758882	2.735758882	2.599377858	2.599377861	$1.363810240 \times 10^{-1}$

These results show the accuracy of trapezoidal method to solve equation (5.3) since the max $error = 2.045715329 \times 10^{-1}$.

We conclude from our numerical test cases that the Taylor expansion method is more efficient than the Trapezoidal method.

Conclusions

In this thesis, some analytical and numerical methods for solving fuzzy Fredholm integral equation of the second kind are presented. The analytical methods are: Fuzzy Laplace method, Homotopy analysis method (HAM), Adomain decomposition method (ADM) and Fuzzy differential transformation method (FDTM). The numerical methods include: Taylor expansion method and Trapezoidal method.

In addition, the numerical methods were implemented in a form of algorithms to solve some cases using MAPLE software. Numerical results have shown to be in a closed agreement with the analytical ones.

However, for the numerical methods the Taylor expansion method seems to be more accurate than the Trapezoidal method according to our numerical test cases.

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Appendix

Maple Code for example (5.1)

$restart$:
$Algorithm\,(5.1)$;
$with(LinearAlgebra)$:
$with(linalg)$:
$with(Tolerances)$:

$\text{lambda} := 1;\, a := 0;\, b := 2;\, z := \frac{1}{2};\, N := 1;\, m := 1$

$f_L(t,r) := \frac{1}{2}\cdot(r+1)\cdot t\,;\, f_{L0} := subs\big(t=z, f_L(t,r)\big)$

$f_U(t,r) := \frac{1}{2}\cdot(3-r)\cdot t;\, f_{U0} := subs\big(t=z, f_U(t,r)\big)$

$k(s,t) := \frac{1}{4}\cdot t$

for p **from** 0 **to** N **by** 1 **do**;
for q **from** 0 **to** N **by** 1 **do**;
$d[0] := subs(t=z, k(s,t))$;
$d[1] := subs(t=z, diff(k(s,t), t\$1))$;
$w_{p,q} := \frac{1}{q!}\cdot\big(int\big(\mathrm{d[p]}\cdot(s-z)^q, s=0\,..2\big)\big)$;
$print\big(w_{p,q}\big)$;
od;
od;

$W(1,1) := Matrix\big(2, 2, \big[w_{0,0}-1, w_{0,1}, w_{1,0}, w_{1,1}-1\big]\big)$

$w := Matrix(2)$

$W(2,2) := W(1,1)$;
$W := Matrix(4, 4, [[W(1,1), w], [w, W(2,2)]])$;
$WI := MatrixInverse(W)$;

$f_{L[0]} := subs\left(t = z, f_L(t, r)\right);$
$f_{L[1]} := subs\left(t = z, diff\left(f_L(t, r), t\$1\right)\right);$
$f_{U[0]} := subs\left(t = z, f_U(t, r)\right);$
$f_{U[1]} := subs\left(t = z, diff\left(f_U(t, r), t\$1\right)\right);$

$$F\left(\frac{1}{2}, r\right) := Matrix\left(4, 1, \left[-f_{L[0]}, -f_{L[1]}, -f_{U[0]}, -f_{U[1]}\right]\right)$$

$$G := MatrixInverse(W).F\left(\frac{1}{2}, r\right)$$

$$g_{L[0]} := \frac{1}{0!} \cdot G(1, 1) \cdot (t - z)^0 + \frac{1}{1!} \cdot G(2, 1) \cdot (t - z)^1$$

$$g_{U[0]} := \frac{1}{0!} \cdot G(3, 1) \cdot (t - z)^0 + \frac{1}{1!} \cdot G(4, 1) \cdot (t - z)^1$$

$$g_L := subs\left(t = 1.001, g_{L[0]}\right)$$

$$g_U := subs\left(t = 1.001, g_{U[0]}\right)$$

$$plot([x - 1, 3 - x, (x - 1), (3 - x)], x = 0..4, 0..1, style = [patch, patch, point, point], color = [black, black, red, red])$$

$\boldsymbol{t := 1:}$
for r **from** 0.0 **to** 1.0 **by** 0.1 **do**:
$y_L := \mathrm{abs}((r + 1) \cdot t - ((r + 1) \cdot \mathrm{t})):$
$y_U := \mathrm{abs}((3 - r) \cdot t - ((3 - r) \cdot \mathrm{t})):$
$e_L := maximize\left(y_L\right):$
$e_U := maximize\left(y_U\right):$
$e := \max\left(e_L, e_U\right):$
$print\left(r, ` `, e_L, ` `, e_U, ` `, e\right);$
od;

Maple code for example (5.2)

```
restart :
Algorithm (5.2);
with(LinearAlgebra) :
with(linalg) :
with(Tolerances) :
```

$\text{lambda} := 1;\ a := 0;\ b := 1;\ z := \frac{1}{2};\ N := 3;\ m := 1$

$f_L(t, r) := (r + 1)\cdot(\exp(-t) + t - \sin(t));$

$f_{L0} := evalf\big(subs\big(t = z, f_L(t, r)\big)\big)$

$f_U(t, r) := (3 - r)\cdot(\exp(-t) + t - \sin(t));$

$f_{U0} := evalf\big(subs\big(t = z, f_U(t, r)\big)\big)$

$k(s, t) := \frac{1}{2}\cdot\exp(s)\cdot\sin(t)$

for p **from** 0 **to** N **by** 1 **do**:
for q **from** 0 **to** N **by** 1 **do**:

$d[0] := evalf(subs(t = z, k(s, t)))$:

$d[1] := evalf(subs(t = z, diff(k(s, t), t\$1)))$:

$d[2] := evalf(subs(t = z, diff(k(s, t), t\$2)))$:

$d[3] := evalf(subs(t = z, diff(k(s, t), t\$3)))$:

$w_{p,\,q} := \frac{1}{q!}\cdot\big(int\big(\mathrm{d[p]}\cdot(s - z)^q, s = 0..1\big)\big)$:

$print\big(p, ``, q, ``, w_{p,\,q}\big)$:

od:
od:

$W(1, 1) := Matrix\big(4, 4, \big[w_{0,0} - 1, w_{0,1}, w_{0,2}, w_{0,3}, w_{1,0}, w_{1,1} - 1, w_{1,2}, w_{1,3}, w_{2,0}, w_{2,1}, w_{2,2} - 1, w_{2,3}, w_{3,0}, w_{3,1}, w_{3,2}, w_{3,3} - 1\big]\big)$

$w := Matrix(4)$

$W(2, 2) := W(1, 1);$

$W := Matrix(8, 8, [[W(1, 1), w], [w, W(2, 2)]]);$

$WI := MatrixInverse(W);$

$f_{LL[0]} := f_L(t, r);$

$f_{LL[1]} := \left(t = z, diff\left(f_L(t, r), t\$1\right)\right);$

$f_{LL[2]} := t = z, diff\left(f_L(t, r), t\$2\right);$

$f_{LL[3]} := t = z, diff\left(f_L(t, r), t\$3\right);$

$f_{UU[0]} := t = z, f_U(t, r);$

$f_{UU[1]} := t = z, diff\left(f_U(t, r), t\$1\right);$

$f_{UU[2]} := t = z, diff\left(f_U(t, r), t\$2\right);$

$f_{UU[3]} := t = z, diff\left(f_U(t, r), t\$3\right);$

$f_{L[0]} := evalf\left(subs\left(t = z, f_L(t, r)\right)\right);$

$f_{L[1]} := evalf\left(subs\left(t = z, diff\left(f_L(t, r), t\$1\right)\right)\right);$

$f_{L[2]} := evalf\left(subs\left(t = z, diff\left(f_L(t, r), t\$2\right)\right)\right);$

$f_{L[3]} := evalf\left(subs\left(t = z, diff\left(f_L(t, r), t\$3\right)\right)\right);$

$f_{U[0]} := evalf\left(subs\left(t = z, f_U(t, r)\right)\right);$

$f_{U[1]} := evalf\left(subs\left(t = z, diff\left(f_U(t, r), t\$1\right)\right)\right);$

$f_{U[2]} := evalf\left(subs\left(t = z, diff\left(f_U(t, r), t\$2\right)\right)\right);$

$f_{U[3]} := evalf\left(subs\left(t = z, diff\left(f_U(t, r), t\$3\right)\right)\right);$

$$F(\mathrm{z}, r) := Matrix\left(8, 1, \left[-f_{L[0]}, -f_{L[1]}, -f_{L[2]}, -f_{L[3]}, -f_{U[0]}, -f_{U[1]}, -f_{U[2]}, -f_{U[3]}\right]\right)$$

$$G := MatrixInverse(W).F(\mathrm{z}, r)$$

$$g_{L[0]} := \frac{1}{0!} \cdot G(1, 1) \cdot (t - z)^0 + \frac{1}{1!} \cdot G(2, 1) \cdot (t - z)^1 + \frac{1}{2!} \cdot G(3, 1) \cdot (t - z)^2 + \frac{1}{3!} \cdot G(4, 1) \cdot (t - z)^3$$

$$g_{U[0]} := \frac{1}{0!} \cdot G(5, 1) \cdot (t - z)^0 + \frac{1}{1!} \cdot G(6, 1) \cdot (t - z)^1 + \frac{1}{2!} \cdot G(7, 1) \cdot (t - z)^2 + \frac{1}{3!} \cdot G(8, 1) \cdot (t - z)^3$$

$$g_L := subs\left(t = 1, g_{L[0]}\right)$$

$$g_U := subs\left(t = 1, g_{U[0]}\right)$$

$$plot\left(\left[\frac{1}{\exp(-1)+1}\cdot x-1, 3-\frac{1}{\exp(-1)+1}\cdot x, \left(\frac{1}{1.36602049107033}\cdot x-1\right), \left(3-\frac{1}{1.36602049107033}\cdot x\right)\right], x = 1..4.5, 0..1, style=[patch, patch, point, point], color=[black, black, red, red]\right)$$

```
t := 1 :
for r from 0.0 to 1.0 by 0.1 do:
z1 := evalf((r + 1)·(exp(-t) + t)) :
z2 := evalf((3 - r)·(exp(-t) + t)) :
q1 := evalf(g_L[0]) :
q2 := evalf(g_U[0]) :

y_L := abs(z1 - q1) :
y_U := abs(z2 - q2) :
e_L := maximize(y_L) :
e_U := maximize(y_U) :
e := max(e_L, e_U) :
print(r, ` `, z1, ` `, z2, ` `, q1, ` `, q2, ` `, e );
od:
```

Maple Code for example (5.3)

restart :

Algorithm (5.3) :

with student :

lambda := 1; a := 0.0; b := 2; n := 51;

$$h := \frac{(b-a)}{n}$$

$$f_{L0}(t, r) := \frac{1}{2.0}\cdot(r+1)\cdot t;$$

$$f_{U0}(t, r) := \frac{1}{2.0}\cdot(3-r)\cdot t;$$

$$f_{L1}(t, r) := 0.9815093968\cdot(1\cdot r+1)\cdot t;$$

$$f_{U1}(t, r) := 0.9815093968\cdot(3.000000001-1\cdot r)\cdot t;$$

$$k(s, t) := \frac{1}{4.0}\cdot t;$$

for *i* **from** 1.0 **to** 49.0 **by** 1 **do**:

$t[i] := \mathrm{a} + i \cdot h;$

$print(i, \text{` `}, t[i]);$

od:

$$S_{L50}(r) := evalf\Bigg((\mathrm{h}) \cdot \bigg(\frac{\big(subs\big(t=0, f_{L1}(t,r)\big) + subs\big(t=2, f_{L1}(t,r)\big)\big)}{2} + subs\big(t = 0.01960784314, f_{L1}(t,r)\big) + subs\big(t = 0.03921568628, f_{L1}(t,r)\big) + subs\big(t = 0.05882352942, f_{L1}(t,r)\big) + subs\big(t = 0.07843137256, f_{L1}(t,r)\big) + subs\big(t = 0.09803921570, f_{L1}(t,r)\big) + subs\big(t = 0.1176470588, f_{L1}(t,r)\big) + subs\big(t = 0.1372549020, f_{L1}(t,r)\big) + subs\big(t = 0.1568627451, f_{L1}(t,r)\big) + subs\big(t = 0.1764705883, f_{L1}(t,r)\big) + subs\big(t = 0.1960784314, f_{L1}(t,r)\big) + subs\big(t = 0.2156862745, f_{L1}(t,r)\big) + subs\big(t = 0.2352941177, f_{L1}(t,r)\big) + subs\big(t = 0.2549019608, f_{L1}(t,r)\big) + subs\big(t = 0.2745098040, f_{L1}(t,r)\big) + subs\big(t = 0.2941176471, f_{L1}(t,r)\big) + subs\big(t = 0.3137254902, f_{L1}(t,r)\big) + subs\big(t = 0.3333333334, f_{L1}(t,r)\big) + subs\big(t = 0.3529411765, f_{L1}(t,r)\big) + subs\big(t = 0.3725490197, f_{L1}(t,r)\big) + subs\big(t = 0.3921568628, f_{L1}(t,r)\big) + subs\big(t = 0.4117647059, f_{L1}(t,r)\big) + subs\big(t = 0.4313725491, f_{L1}(t,r)\big) + subs\big(t = 0.4509803922, f_{L1}(t,r)\big) + subs\big(t = 0.4705882354, f_{L1}(t,r)\big) + subs\big(t = 0.4901960785, f_{L1}(t,r)\big) + subs\big(t = 0.5098039216, f_{L1}(t,r)\big) + subs\big(t = 0.5294117648, f_{L1}(t,r)\big) + subs\big(t = 0.5490196079, f_{L1}(t,r)\big) + subs\big(t = 0.5686274511, f_{L1}(t,r)\big) + subs\big(t = 0.5882352942, f_{L1}(t,r)\big) + subs\big(t = 0.6078431373, f_{L1}(t,r)\big) + subs\big(t = 0.6274509805, f_{L1}(t,r)\big) + subs\big(t = 0.6470588236, f_{L1}(t,r)\big) + subs\big(t = 0.6666666668, f_{L1}(t,r)\big) + subs\big(t = 0.6862745099, f_{L1}(t,r)\big) + subs\big(t = 0.7058823530, f_{L1}(t,r)\big) + subs\big(t = 0.7254901962, f_{L1}(t,r)\big) + subs\big(t = 0.7450980393, f_{L1}(t,r)\big) + subs\big(t = 0.7647058825, f_{L1}(t,r)\big) + subs\big(t = 0.7843137256, f_{L1}(t,r)\big) + subs\big(t = 0.8039215687, f_{L1}(t,r)\big) + subs\big(t = 0.8235294119, f_{L1}(t,r)\big) + subs\big(t = 0.8431372550, f_{L1}(t,r)\big) + subs\big(t = 0.8627450982, f_{L1}(t,r)\big) + subs\big(t = 0.8823529413, f_{L1}(t,r)\big) + subs\big(t = 0.9019607844, f_{L1}(t,r)\big) + subs\big(t = 0.9215686276, f_{L1}(t,r)\big) + subs\big(t = 0.9411764707, f_{L1}(t,r)\big) + subs\big(t = 0.9607843139, f_{L1}(t,r)\big) \bigg) \Bigg);$$

$$S_{U50}(r) := evalf\Big((\mathrm{h}) \cdot \Big(\frac{\big(subs\big(t=0, f_{U1}(t,r)\big) + subs\big(t=2, f_{U1}(t,r)\big)\big)}{2} + subs\big(t = 0.01960784314, f_{U1}(t,r)\big) + subs\big(t = 0.03921568628, f_{U1}(t,r)\big) + subs\big(t = 0.05882352942, f_{U1}(t,r)\big) + subs\big(t = 0.07843137256, f_{U1}(t,r)\big) + subs\big(t = 0.09803921570, f_{U1}(t,r)\big) + subs\big(t = 0.1176470588, f_{U1}(t,r)\big) + subs\big(t = 0.1372549020, f_{U1}(t,r)\big) + subs\big(t = 0.1568627451, f_{U1}(t,r)\big) + subs\big(t = 0.1764705883, f_{U1}(t,r)\big) + subs\big(t = 0.1960784314, f_{U1}(t,r)\big) + subs\big(t = 0.2156862745, f_{U1}(t,r)\big) + subs\big(t = 0.2352941177, f_{U1}(t,r)\big) + subs\big(t = 0.2549019608, f_{U1}(t,r)\big) + subs\big(t = 0.2745098040, f_{U1}(t,r)\big) + subs\big(t = 0.2941176471, f_{U1}(t,r)\big) + subs\big(t = 0.3137254902, f_{U1}(t,r)\big) + subs\big(t = 0.3333333334, f_{U1}(t,r)\big) + subs\big(t = 0.3529411765, f_{U1}(t,r)\big) + subs\big(t = 0.3725490197, f_{U1}(t,r)\big) + subs\big(t = 0.3921568628, f_{U1}(t,r)\big) + subs\big(t = 0.4117647059, f_{U1}(t,r)\big) + subs\big(t = 0.4313725491, f_{U1}(t,r)\big) + subs\big(t = 0.4509803922, f_{U1}(t,r)\big) + subs\big(t = 0.4705882354, f_{U1}(t,r)\big) + subs\big(t = 0.4901960785, f_{U1}(t,r)\big) + subs\big(t = 0.5098039216, f_{U1}(t,r)\big) + subs\big(t = 0.5294117648, f_{U1}(t,r)\big) + subs\big(t = 0.5490196079, f_{U1}(t,r)\big) + subs\big(t = 0.5686274511, f_{U1}(t,r)\big) + subs\big(t = 0.5882352942, f_{U1}(t,r)\big) + subs\big(t = 0.6078431373, f_{U1}(t,r)\big) + subs\big(t = 0.6274509805, f_{U1}(t,r)\big) + subs\big(t = 0.6470588236, f_{U1}(t,r)\big) + subs\big(t = 0.6666666668, f_{U1}(t,r)\big) + subs\big(t = 0.6862745099, f_{U1}(t,r)\big) + subs\big(t = 0.7058823530, f_{U1}(t,r)\big) + subs\big(t = 0.7254901962, f_{U1}(t,r)\big) + subs\big(t = 0.7450980393, f_{U1}(t,r)\big) + subs\big(t = 0.7647058825, f_{U1}(t,r)\big) + subs\big(t = 0.7843137256, f_{U1}(t,r)\big) + subs\big(t = 0.8039215687, f_{U1}(t,r)\big) + subs\big(t = 0.8235294119, f_{U1}(t,r)\big) + subs\big(t = 0.8431372550, f_{U1}(t,r)\big) + subs\big(t = 0.8627450982, f_{U1}(t,r)\big) + subs\big(t = 0.8823529413, f_{U1}(t,r)\big) + subs\big(t = 0.9019607844, f_{U1}(t,r)\big) + subs\big(t = 0.9215686276, f_{U1}(t,r)\big) + subs\big(t = 0.9411764707, f_{U1}(t,r)\big) + subs\big(t = 0.9607843139, f_{U1}(t,r)\big)\Big)\Big);$$

$$\mathrm{l} := \mathrm{int}\big(\mathrm{S}_{L50}(\mathrm{r}), \mathrm{s} = 0\,..2\big);$$

$$\mathrm{ll} := \mathrm{int}\big(\mathrm{S}_{U50}(\mathrm{r}), \mathrm{s} = 0\,..2\big);$$

$$S_{L51}(t,r) := evalf\big(f_{L0}(t,r) + \mathrm{k}(\mathrm{s},\mathrm{t}) \cdot l\big);$$

$$S_{U51}(t,r) := evalf\big(f_{U0}(t,r) + \mathrm{k}(\mathrm{s},\mathrm{t}) \cdot ll\big);$$

$$plot\left(\left[\frac{1}{1}\cdot x-1, 3-\frac{1}{1}\cdot x, \frac{1}{0.9815094158}\cdot(x-0.9815094158\cdot 1), \frac{1}{0.9815094158\cdot 1}\cdot(0.9815094158\cdot 3.000000001\cdot 1-x)\right], x=0..4, 0..1, style=[patch, patch, point, point], color=[black, black, red, red]\right)$$

```
t := 1 :
for r from 0.0 to 1.0 by 0.1 do:
z1 := (r + 1)·t :
q1 := (3 − r)·t :
z2 := (0.9815094158· r + 0.9815094158)·t :
q2 := (0.9815094158·3.000000001 − 0.9815094158· r)·t :
y_L := abs(z1 − z2) :
y_U := abs(q1 − q2) :
e_L := maximize(y_L) :
e_U := maximize(y_U) :
e := max(e_L, e_U) :
print(z1, ` `, q1, ` `, z2, ` `, q2, ` `, e);
od:
```

Maple Code for example (5.4)

```
restart :
with student :
with(Tolerances) :
lambda := 1; a := 0; b := 1; n := 51;

h := evalf((b − a)/n)
f_L0(t, r) := (r + 1)·(exp(−t) + t − sin(t));
f_U0(t, r) := (3 − r)·(exp(−t) + t − sin(t));
f_L1(t, r) := ((r + 1)·(exp(−t) + t) + (−0.0810372705·r
    − 0.0810372705)·sin(t));
f_U1(t, r) := ((3 − r)·(exp(−t) + t) + (−0.243111808
    + 0.0810372705·r)·sin(t));
k(s, t) := 1/2·exp(s)·sin(t);

for i from 1.0 to 49.0 by 1 do:
m[i] := a + i·h;
print(i, ` `, m[i]);
od:
```

$$
\begin{aligned}
S_{L50}(r) := evalf\Bigg(& (\mathrm{h}) \\
& \cdot \left(\frac{\left(subs\left(t=0, f_{L1}(t,r)\right) + subs\left(t=1, f_{L1}(t,r)\right)\right)}{2} + subs\left(t \right.\right. \\
& = 0.01960784314, f_{L1}(t,r)\big) + subs\left(t = 0.03921568628, f_{L1}(t,r)\right) \\
& + subs\left(t = 0.05882352942, f_{L1}(t,r)\right) + subs\big(t = 0.07843137256, \\
& f_{L1}(t,r)\big) + subs\left(t = 0.09803921570, f_{L1}(t,r)\right) + subs\big(t \\
& = 0.1176470588, f_{L1}(t,r)\big) + subs\left(t = 0.1372549020, f_{L1}(t,r)\right) \\
& + subs\left(t = 0.1568627451, f_{L1}(t,r)\right) + subs\big(t = 0.1764705883, \\
& f_{L1}(t,r)\big) + subs\left(t = 0.1960784314, f_{L1}(t,r)\right) + subs\big(t \\
& = 0.2156862745, f_{L1}(t,r)\big) + subs\left(t = 0.2352941177, f_{L1}(t,r)\right) \\
& + subs\left(t = 0.2549019608, f_{L1}(t,r)\right) + subs\big(t = 0.2745098040, \\
& f_{L1}(t,r)\big) + subs\left(t = 0.2941176471, f_{L1}(t,r)\right) + subs\big(t \\
& = 0.3137254902, f_{L1}(t,r)\big) + subs\left(t = 0.3333333334, f_{L1}(t,r)\right) \\
& + subs\left(t = 0.3529411765, f_{L1}(t,r)\right) + subs\big(t = 0.3725490197, \\
& f_{L1}(t,r)\big) + subs\left(t = 0.3921568628, f_{L1}(t,r)\right) + subs\big(t \\
& = 0.4117647059, f_{L1}(t,r)\big) + subs\left(t = 0.4313725491, f_{L1}(t,r)\right) \\
& + subs\left(t = 0.4509803922, f_{L1}(t,r)\right) + subs\big(t = 0.4705882354, \\
& f_{L1}(t,r)\big) + subs\left(t = 0.4901960785, f_{L1}(t,r)\right) + subs\big(t \\
& = 0.5098039216, f_{L1}(t,r)\big) + subs\left(t = 0.5294117648, f_{L1}(t,r)\right) \\
& + subs\left(t = 0.5490196079, f_{L1}(t,r)\right) + subs\big(t = 0.5686274511, \\
& f_{L1}(t,r)\big) + subs\left(t = 0.5882352942, f_{L1}(t,r)\right) + subs\big(t \\
& = 0.6078431373, f_{L1}(t,r)\big) + subs\left(t = 0.6274509805, f_{L1}(t,r)\right) \\
& + subs\left(t = 0.6470588236, f_{L1}(t,r)\right) + subs\big(t = 0.6666666668, \\
& f_{L1}(t,r)\big) + subs\left(t = 0.6862745099, f_{L1}(t,r)\right) + subs\big(t \\
& = 0.7058823530, f_{L1}(t,r)\big) + subs\left(t = 0.7254901962, f_{L1}(t,r)\right) \\
& + subs\left(t = 0.7450980393, f_{L1}(t,r)\right) + subs\big(t = 0.7647058825, \\
& f_{L1}(t,r)\big) + subs\left(t = 0.7843137256, f_{L1}(t,r)\right) + subs\big(t \\
& = 0.8039215687, f_{L1}(t,r)\big) + subs\left(t = 0.8235294119, f_{L1}(t,r)\right) \\
& + subs\left(t = 0.8431372550, f_{L1}(t,r)\right) + subs\big(t = 0.8627450982, \\
& f_{L1}(t,r)\big) + subs\left(t = 0.8823529413, f_{L1}(t,r)\right) + subs\big(t \\
& = 0.9019607844, f_{L1}(t,r)\big) + subs\left(t = 0.9215686276, f_{L1}(t,r)\right) \\
& + subs\left(t = 0.9411764707, f_{L1}(t,r)\right) + subs\big(t = 0.9607843139, \\
& f_{L1}(t,r)\big)\Big)\Bigg)
\end{aligned}
$$

$$S_{U50}(r) := evalf\Big((\mathrm{h}) \cdot \Big(\frac{\big(subs(t=0, f_{U1}(t,r)) + subs(t=1, f_{U1}(t,r))\big)}{2} + subs(t = 0.01960784314, f_{U1}(t,r)) + subs(t = 0.03921568628, f_{U1}(t,r)) + subs(t = 0.05882352942, f_{U1}(t,r)) + subs(t = 0.07843137256, f_{U1}(t,r)) + subs(t = 0.09803921570, f_{U1}(t,r)) + subs(t = 0.1176470588, f_{U1}(t,r)) + subs(t = 0.1372549020, f_{U1}(t,r)) + subs(t = 0.1568627451, f_{U1}(t,r)) + subs(t = 0.1764705883, f_{U1}(t,r)) + subs(t = 0.1960784314, f_{U1}(t,r)) + subs(t = 0.2156862745, f_{U1}(t,r)) + subs(t = 0.2352941177, f_{U1}(t,r)) + subs(t = 0.2549019608, f_{U1}(t,r)) + subs(t = 0.2745098040, f_{U1}(t,r)) + subs(t = 0.2941176471, f_{U1}(t,r)) + subs(t = 0.3137254902, f_{U1}(t,r)) + subs(t = 0.3333333334, f_{U1}(t,r)) + subs(t = 0.3529411765, f_{U1}(t,r)) + subs(t = 0.3725490197, f_{U1}(t,r)) + subs(t = 0.3921568628, f_{U1}(t,r)) + subs(t = 0.4117647059, f_{U1}(t,r)) + subs(t = 0.4313725491, f_{U1}(t,r)) + subs(t = 0.4509803922, f_{U1}(t,r)) + subs(t = 0.4705882354, f_{U1}(t,r)) + subs(t = 0.4901960785, f_{U1}(t,r)) + subs(t = 0.5098039216, f_{U1}(t,r)) + subs(t = 0.5294117648, f_{U1}(t,r)) + subs(t = 0.5490196079, f_{U1}(t,r)) + subs(t = 0.5686274511, f_{U1}(t,r)) + subs(t = 0.5882352942, f_{U1}(t,r)) + subs(t = 0.6078431373, f_{U1}(t,r)) + subs(t = 0.6274509805, f_{U1}(t,r)) + subs(t = 0.6470588236, f_{U1}(t,r)) + subs(t = 0.6666666668, f_{U1}(t,r)) + subs(t = 0.6862745099, f_{U1}(t,r)) + subs(t = 0.7058823530, f_{U1}(t,r)) + subs(t = 0.7254901962, f_{U1}(t,r)) + subs(t = 0.7450980393, f_{U1}(t,r)) + subs(t = 0.7647058825, f_{U1}(t,r)) + subs(t = 0.7843137256, f_{U1}(t,r)) + subs(t = 0.8039215687, f_{U1}(t,r)) + subs(t = 0.8235294119, f_{U1}(t,r)) + subs(t = 0.8431372550, f_{U1}(t,r)) + subs(t = 0.8627450982, f_{U1}(t,r)) + subs(t = 0.8823529413, f_{U1}(t,r)) + subs(t = 0.9019607844, f_{U1}(t,r)) + subs(t = 0.9215686276, f_{U1}(t,r)) + subs(t = 0.9411764707, f_{U1}(t,r)) + subs(t = 0.9607843139, f_{U1}(t,r))\Big)\Big)$$

$$\mathrm{l} := \mathrm{int}\big(\exp(\mathrm{s}) \cdot \mathrm{S}_{L50}(\mathrm{r}), \mathrm{s} = 0..1\big);$$

$$\mathrm{ll} := \mathrm{int}\big(\exp(\mathrm{s}) \cdot \mathrm{S}_{U50}(\mathrm{r}), \mathrm{s} = 0..1\big);$$

$$S_{L51}(t,r) := evalf\Big(f_{L0}(t,r) + \frac{1}{2} \cdot \sin(t) \cdot l\Big);$$

$$S_{U51}(t,r) := evalf\left(f_{U0}(t,r) + \frac{1}{2}\cdot\sin(t)\cdot ll\right);$$

$$plot\left(\left[\frac{1}{\exp(-1)+1}\cdot x - 1, 3 - \frac{1}{\exp(-1)+1}\cdot x, \frac{\mathrm{x} - \exp(-1) - 1 + 0.0810372705\cdot\sin(1)}{\exp(-1) + 1 - 0.0810372705\cdot\sin(1)}, \frac{\mathrm{x} - 3\cdot\exp(-1) - 3 + 0.243111808\cdot\sin(1)}{0.0810372705\cdot\sin(1) - \exp(-1) - 1}\right], x = 1..4.5, 0..1, style = [patch, patch, point, point], color = [black, black, red, red]\right)$$

$\boldsymbol{t} := \mathbf{1.0:}$
for r **from** 0.0 **to** 1.0 **by** 0.1 **do**:
$z1 := \mathrm{evalf}((r+1)\cdot(\exp(-t)+t));$
$q1 := evalf((3-r)\cdot(\exp(-t)+t));$
$z2 := evalf(((r+1)\cdot(\exp(-t)+t) + (-0.0810372705\cdot \mathrm{r} - 0.0810372705)\cdot\sin(t))):$
$q2 := evalf(((3-r)\cdot(\exp(-t)+t) + (-0.243111808 + 0.0810372705\cdot r)\cdot\sin(t))):$
$y_L := \mathrm{abs}(\mathrm{z1} - \mathrm{z2}):$
$y_U := \mathrm{abs}(\mathrm{q1} - \mathrm{q2}):$
$e_L := maximize(y_L):$
$e_U := maximize(y_U):$
$e := \max(e_L, e_U):$
$print(r, ` `, z1, ` `, q1, ` `, z2, ` `, q2, ` `, e);$
od:

جامعة النجاح الوطنية
كلية الدراسات العليا

معادلة فريدهولم التكاملية الضبابية من النوع الثاني

إعداد
منى شاهر يوسف عماوي

إشراف
أ.د ناجي قطناني

قدمت هذه الرسالة استكمالا لمتطلبات الحصول على درجة الماجستر في الرياضيات بكلية الدراسات العليا في جامعة النجاح الوطنية، نابلس–فلسطين.

2014

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الملخص

معادلات فرديهولم التكاملية الضبابية من النوع الثاني تلقت اهتماماً كبيراً نظراً لأهمية هذه المعادلات في الدراسات المرتبطة مع التطبيقات في الرياضيات الفيزيائية والنظم المالية والاقتصادية الضبابية.

بعد أن تناولنا المفاهيم الأساسية للمعادلات التكاملية الضبابية، قمنا باستقصاء بعض الطرق التحليلية والعددية لحل معادلة فريدهولم التكاملية الضابية من النوع الثاني. هذه الطرق التحليلة شملت: طريقة تحويل لابلاس الضبابية، طريقة هوموتوبي التحليلة، طريقة أدومين التحليلية، طريقة التحويل التفاضلية الضبابية.

الطرق العددية التي قمنا بتنفيذها هي: طريقة تايلور التوسعية، طريقة شبه المنحرف. بعد ذلك قمنا بتنفيذ بعض الأمثلة باستخدام تلك الطرق العددية. وأجرينا مقارنة بين النتائج التحليلية والنتائج التقريبية. النتائج التقريبية أظهرت دقتها وقربها من النتائج التحليلة.