An-Najah National University

Faculty of Graduate Studies

Cone Metric Spaces

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الاهداء

الى من بلغ الرسالة وأدى الامانة ونصح الامة ، الى" نبي الرحمة ونور العالمين "

الى شمعة البيت ، الى من أرضعتني الطموح ، الى من اجتازت خيول دعواتها حدود السماء ...

" امي الحبيبة "

الى من احمل اسمه بكل فخر ، ارجو الله ان يمد في عمره ليرى ثمارا قد حان قطافها بعد طول انتظار ... "والدي العزيز "

الى من امضيت بجانبها اجمل سنوات حياتي ، الى من كانت خير سند لي ومعين ...

زوجتي "ام قاسم "

الى شق روحي ورفاقي عند الصعاب ، بهم أكبر وعليهم اعتمد ، الى من بوجودهم اكتسب القوة ، ورسمت معهم احلا ذكرياتي ، الى سندي عند الصعاب... " اخوتي واختي"

الى شمسي وقمري ، ومن ساندوني ووقفوا بجانبي في احلك ظروفي ...

عمتاي "خديجة و هدى "

الى من تتلمذت على ايديهم ، الى كل من علمني حرفا في مسيرتي التعليمية...

" اساتذتي الكرام"

اليكم جميعا اهدي عملي هذا سائلا المولى عز وجل ان يوفقني في اكمال مسيرتي التعليمية

شكر وتقدير

عرفانا مني بجميل من كان له فضل علي ، فإنني اشكر استاذي الفاضل الدكتور عبد الله حكواتي ، الذي اشرف على هذه الرسالة ، ومهد لي الطريق بمعرفته الواسعة ، وعلمه النافع الى ان اتم هذا العمل المتواضع ، كما اشكره على سعة صدره وجهده المتواصل في سبيل اتمام هذا العمل واخراجه الى النور .

واتفضل بجزيل الشكر الى اعضاء لجنة المناقشة لتفضلهم بقراءة هذه الرسالة وتقييمها واعطاء ملحوظاتهم عليها لتصحيح أي خطأ فيها وبيان جوانب القوة .

كما اشكر المجلة العلمية (JNAA) على قبولهم لنشر ورقة البحث بعنوان

(Metrizability Of Cone Metric Spaces Via Renorming The Banach Spaces)

الإقرار

أنا الموقع/ ـة أدناه, مقدم/ ـة الرسالة التي تحمل العنوان :

Cone Metric Spaces

أقر بأن ما اشتملت عليه هذه الرسالة انما هي نتاج جهدي الخاص, باستثناء ما تمت الاشارة اليه حيثما ورد, وإن هذه الرسالة ككل, او أي جزء منها لم يقدم من قبل لنيل أية درجة أو لقب علمي أو بحثي لدى أية مؤسسة تعليمية أو بحثية أخرى.

Declaration

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

Student's Name: **اسم الطالب:**

Signature: **التوقيع :**

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Abstract

Cone metric spaces were introduced in [1] by means of partially ordering real Banach spaces by specified cones. In [4] and [8] , the nation of cone – normed spaces was introduced. cone- metric spaces, and hence, cone- normed spaces were shown to be first countable topological spaces. The reader may consult [5] for this development. In [6], it was shown that, in a sense, cone- metric spaces are not, really, generalizations of metric spaces. This was the motive to do further investigations.

Now, we put things in order.

1. Definition:[1] Let $(E, \|\cdot\|)$ be a real Banach space and P a subset of E then P is called a cone if :

   (a) P is closed, convex, nonempty, and $P \neq \{0\}$.

   (b) $a,b \in \mathbb{R}$; $a,b \geq 0$; $x, y \in P \Rightarrow ax+by \in P$.

   (c) $x \in P$ and $-x \in P \Rightarrow x = 0$.

2. Example: [13] Let $E= \ell^1$, the absolutely summable real sequences. Then the set $P = \{x \in E : x_n \geq 0 , \forall n\}$ is a cone in E.

In our project, we will attempt to enforce the feeling that cone metric spaces are not real generalization of metric spaces by the necessary theory and examples. In the meantime, we will keep it conceivable to arrive at generalization aspects.

Preface

Recently, the concept of cone metric space is introduced and some fixed point theorems for contractive mappings in a cone metric space were established. Indeed, the authors there replace the real numbers $\mathbb{R}$ by ordering a Banach space E to define cone metric space.

After that, series of articles about cone metric spaces started to appear. Some of those articles dealt with fixed point theorems in those spaces and some other with the structure of spaces themselves.

Convergent and Cauchy sequences, complete spaces, normed spaces, metric spaces and others are studied in a new way in cone metric space.

Topological questions were answered in cone metric spaces, where it was proved that cone metric spaces are: topological space, first countable space, Hausdorff space and T_4 space.

One of the important questions that will appear is: " are cone metric spaces real generalizations of metric spaces or they are equivalent? " Recently, this question has been investigated by many authors by showing that cone metric spaces are metrizable and defining the equivalent metric using a variety of approaches.

So our belief is that a cone metric space is really a metric space and every theorem in metric spaces is valid for cone metric space automatically. This enforces the feeling that cone metric spaces are not real generalization of metric spaces. In the meantime, we will keep it conceivable to arrive at generalization aspects. So we suggested a section which is titled by " On generalization possibilities ".

Finally, we define new concepts of measure theory in the sense of cone metric space.

This thesis consists of five chapters. Each chapter is divided into sections. A number like 2.1.3 indicates item (definition, theorem, proposition, remark, lemma or example) number 3 in section 1 of chapter 2. Each chapter begins with a clear statement of the pertinent definitions and theorems together with illustrative and descriptive material. At the end of this thesis we present a collection of references.

In chapter (1) we introduce the basic results and definitions which shall be needed in the following chapters. The topics include normed space, Banach space, cone, properties and examples of cones, cone metric spaces, convergent sequence, Cauchy sequence, complete cone metric space, cone normed space and cone Banach space.

Chapter (2) will be devoted to give an introduction to fundamental ideas of metrizability of cone metric spaces. We will start by introducing a metrizable space and presenting some efforts of some authors in showing that cone metric spaces are metrizable spaces. Finally, we give our contributions in this direction. We would like to mention here that this contribution is accepted for publication in the journal of nonlinear analysis and application (**JNAA**).

In chapter (3) some topological concepts and definitions are generalized to cone metric spaces. The topics includes distance between two nonempty subsets, bounded subsets, sequentially closed subsets, normal space, T_4 space, continuous mappings, sequentially continuous maps, c-net subsets, totally bounded subsets, Lebesgue element, compact and sequentially compact subsets. Also, it was proved that cone metric spaces are: topological spaces, first countable space, Hausdorff space and T_4 space.

Chapter (4) has two purposes. First, we review some fixed point theorems of contractive mappings in cone metric spaces. Second, we obtain some examples in cone metric spaces that some properties are incorrect in these spaces but hold in ordinary case (metric space) and conversely. First example states that comparison test does not hold in cone metric spaces, the second example is for normal cones in which we can find two members f, g of the cone such that $f \leq g$ but $\| f \| \geq \| g \|$, and the last example is a contractive mapping on a cone metric space but not contractive in the Euclidean metric space.

In chapter (5) we shall review the theory of Lebesgue measure, Lebesgue integral and Lebesgue integrable functions. Finally, we define new concepts of measure theory in sense of cone metric space.

Chapter One

Preliminaries

This chapter contains some definitions and basic theorems about normed space, Banach space, cone, normal cones, regular cones, minihedral and strongly minihedral cones, cone metric space, convergent sequences, Cauchy sequences, completeness and cone normed spaces.

Definition 1.1.1: [22] A real **normed space** is a real vector space X together with a map $\|\cdot\|: X \longrightarrow \mathbb{R}$, called the norm, such that:

i) $\|x\| \geq 0$, $\forall x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

ii) $\|\alpha x\| = |\alpha|\|x\|$, $\forall x \in X$ and $\alpha \in \mathbb{R}$.

iii) $\|x+y\| \leq \|x\|+\|y\|$, $\forall x, y \in X$.

Definition 1.1.2: [22] A complete normed spaces is called a **Banach space**.

Definition 1.1.3: [1] Let E be a real Banach space with norm $\|\cdot\|$.

A nonempty convex closed subset $P \subset E$ is called a **cone** if it satisfies:

i) $P \neq \{0\}$.

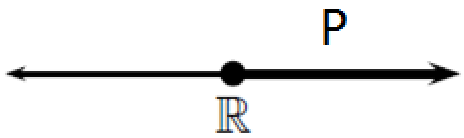
ii) $0 \leq a , b \in \mathbb{R}$ and $x, y \in P$ imply that $ax+by \in P$.

iii) $x \in P$ and $-x \in P$ imply that $x = 0$.

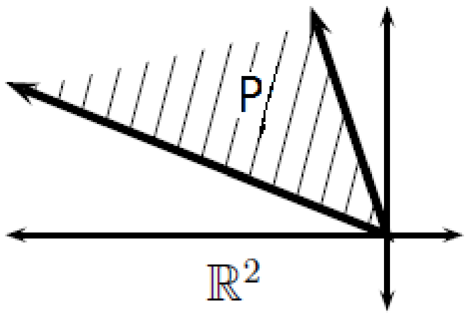
The space E can be partially ordered by the cone $P \subset E$ as follows: $x \leq y$ if and only if $y-x \in P$. We write $x \ll y$ (x is away behind y) if: $y-x \in P^{\circ}$, where P° denotes the interior of P. Also, $x < y$ means that: $x \leq y$ but $x \neq y$.

Some examples of cones: [13]

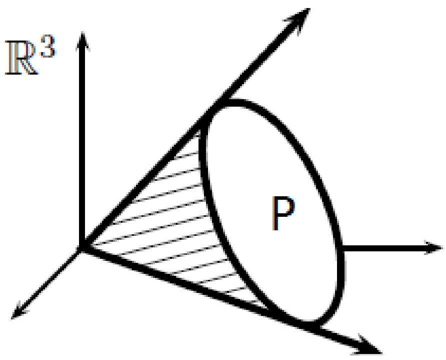
1) In $\mathbb{R}$, the nonnegative numbers form a cone.



2) In $\mathbb{R}^2$, any wedge which extends to infinity from the origin is a cone.



3) Let E = $\mathbb{R}^3$, then P = $\{(x_1, x_2, x_3) \in E \,;\, x_i \geq 0\}$ is a cone.



4) Let E = $\mathbb{R}^n$, then P= $\{(x_1, x_2, \dots, x_n) \in E \,;\, x_i \geq 0\}$ is a cone.

5) In ℓ^p spaces including (ℓ^∞), the set P =$\{x_n \in \ell^p : x_n \geq 0, \forall n\}$ is a cone.

6) In E= C[0 ,1] with the supremum norm, the set P= $\{f \in E : f \geq 0\}$ is a cone.

Definition 1.1.4: [24]A cone P in (E , ‖·‖) is called:

(N)**Normal**: If there exists a constant k > 0 such that:

$0 \leq x \leq y$ implies that $\|x\| \leq k\|y\|$.

The least positive integer k is called the normal constant of P, we will see that there are no cones with constant k < 1.

(R)**Regular**: If every increasing sequence which is bounded above is convergent.

That is if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq ... \leq y$ for some y ∈ E then there exists x ∈ E such that : $\lim_{n\to\infty}\|x_n - x\| = 0$. Equivalently; the cone P is regular if and only if every decreasing sequence which is bounded below is convergent.

Regular cones are normal and there exist normal cones which are not regular, see [2] and [15].

(M)**Minihedral**: If sup {x, y} exists ∀x, y ∈ E and **strongly minihedral** if every subset of E which is bounded above has a supremum.

(S) **Solid**: If $P^{\circ} \neq \emptyset$.

Proposition1.1.5: [2] There are no normal cones with normal constant k < 1.

Proof: Let P be a normal cone with normal constant k < 1, choose a non-zero element x ∈ P and 0 < ε < 1 such that k < (1- ε) then; (1- ε) x ≤ x but (1- ε) ‖x‖ > k‖x‖ , so this is a contradiction, hence there is no normal cones with normal constant k < 1.

Lemma1.1.6: [2] Every regular cone is normal.

Proof: Let P be a regular cone which is not normal. For each n ≥ 1, choose $t_n, s_n \in P$ such that $t_n - s_n \in P$ and $n^2\|t_n\| < \|s_n\|$. For each n ≥ 1 put $y_n = \frac{t_n}{\|t_n\|}$ and $x_n = \frac{s_n}{\|t_n\|}$. Then: x_n , y_n , $y_n - x_n \in P$. $\|y_n\|=1$ and $n^2 < \|x_n\|$, ∀n ≥ 1. Now, since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}\|y_n\|$ is convergent and P is closed, there is an element y ∈P such that $\sum_{n=1}^{\infty} \frac{1}{n^2} y_n = y$.

Since: $0 \le x_1 \le x_1 + \frac{1}{2^2}x_2 \le x_1 + \frac{1}{2^2}x_2 + \frac{1}{3^2}x_3 \le \ldots \le y$, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}x_n$ is convergent because P is regular. Hence, $\lim_{n\to\infty} \frac{\|x_n\|}{n^2} = 0$.

This is a contradiction.

Theorem1.1.7:[2]For each k >1, there is a normal cone with normal constant M > k.

Proof: Let k >1 is given; consider the real vector space E=$\left\{ ax+b : \text{a}, \text{b} \in R; x \in \left[1-\frac{1}{k}, 1 \right] \right\}$ equipped with the supremum norm and P=$\{ax + b \in E : \text{a} \le 0, \text{b} \ge 0\}$. First, we show that P is regular (and so normal).

Let $\{a_n x + b_n\}_{n\ge 1}$ be an increasing sequence which is bounded above that is, there is an element cx+d ∈E such that:

$a_1x + b_1 \le a_2x + b_2 \le .... \le a_nx + b_n \le .... \le cx + d$

Then $\{a_n\}_{n\ge1}$ and $\{b_n\}_{n\ge1}$ are two sequences in ℝ such that:

$b_1 \le b_2 \le b_3 \le \ldots \le d$, $a_1 \le a_2 \le a_3 \le \ldots \le c$

Thus $\{a_n\}_{n\ge1}$ and $\{b_n\}_{n\ge1}$ are convergent by the monotone convergence theorem.

Let $a_n \to a$, $b_n \to b$, then $ax+b \in P$ and $(a_n x + b_n) \to (ax+b)$

Therefore, P is regular and hence normal (by lemm1.1.6).

So there is $M \geq 1$ such that $0 \leq g \leq f$ implies $\|g\| \leq M\|f\|$, for all $g, f \in E$.

Now, we show that $M > k$.

First, note that $f(x) = -kx + k \in P$, $g(x) = k \in P$ and $f-g \in P$ implies that $0 \leq g \leq f$. Therefore, $k = \|g\| \leq M\|f\| = M$.

On the other hand, if we consider $f(x) = -\left(k + \frac{1}{k}\right)x + k$ and $g(x) = k$ then f , g $\in P$ and $f-g \in P$, also $\|g\| = k$ and $\|f\| = 1 - \frac{1}{k} + \frac{1}{k^2}$.

Thus, $k = \|g\| > k\ \|f\| = k + \frac{1}{k} - 1$. This shows that $M > k$.

Proposition 1.1.8:[7] Every strongly minihedral normal cone is regular.

Proof : Let $P \subset E$ be a strongly minihedral normal cone with normal constant k and $a_1 \leq a_2 \leq a_3 \leq \ldots$ increasing and bounded above in E. Since P is strongly minihedral, one can find $\sup\{a_1, a_2, a_3, \ldots\}$ say a.

Claim: $a_n \to a$ in E.

To prove the claim; let $\varepsilon > 0$ be given, choose $c \gg 0$ such that: $k\|c\| \leq \varepsilon$.

Now, $a-c \ll a$, hence find m such that $a-c \ll a_m \ll a$. Then, $0 < a-a_n \ll a-a_m \ll c$, $\forall n \geq m$. Since a_n is increasing $\|a-a_n\| \leq k\|c\| < \varepsilon$, $\forall n \geq m$. Therefore, $\lim_{n\to\infty} a_n = a$.

Proposition1.1.9 : [15]There exist normal cones which are not regular.

The following example gives illustration.

Example1.1.10:[15] Let $E = C[0 , 1]$ with the supremum norm and $P = \{f \in E : f \geq 0\}$ then P is a normal cone with k =1 which is not regular. This is clear, since the sequence x^n is monotonically decreasing, but not uniformly convergent to 0. Thus, P is not strongly minihedral.

2. Cone metric spaces

In what follows, E is a real Banach space, P a cone in E and the partial ordering are given with respect to P.

Definition1.2.1:[1] Let X be a nonempty set, suppose that the mapping

$D: X\times X \to E$ satisfies the following:

(i) $0 \leq D(x , y)$, $\forall x , y \in X$ and $D(x , y)= 0$ if and only if $x = y$.

(ii)$D(x , y) = D(y, x)$.

(iii)$D(x , y) \leq D(x , z)+D(z , y)$, $\forall$ $x , y, z \in X$.

Then D is called a cone metric on X, and the pair (X , D) is called a **cone metric space**.

Definition1.2.2:[1] Let (X , D) be a cone metric space, $x \in X$ and $\{x_n\}$ be a sequence in X, then:

(i)$\{x_n\}$ is said to be **convergent** to $x \in X$ whenever for every $c \in E$ with $c \gg 0$ there is N such that for $n > N$, $D(x_n , x) \ll c$.

(ii)$\{x_n\}$ is called a **Cauchy sequence** in X whenever for every $c \in E$ with $c \gg 0$ there is N such that for each $n, m > N$, $D(x_n , x_m) \ll c$.

(iii)(X , D) is a **complete** cone metric space if every Cauchy sequence is convergent.

Example1.2.3: [25] Let q > 0, E = l^q , P = { $\{x_n\}_{n\ge1}$ ∈ E: $x_n \ge 0$; ∀n}

Let (X, ρ) be a metric space and D: X×X → E defined by:

D(x, y) = $\left\{\left(\dfrac{\rho(x,y)}{2^n}\right)^{\frac{1}{q}}\right\}_{n\ge1}$ .Then (X , D) is a cone metric space and the normal constant of P is equal to k =1.

Example1.2.4 :[25] Let E = ($C_R[0,\infty)$,‖·‖$_\infty$), P = { f ∈ E : f(x) ≥ 0 }, (X , ρ) be a metric space, D: X×X → E defined by D(x , y)=ρ(x,y)ϕ where ϕ:[0, 1]→ R^+ is continuous then (X , D) is a cone metric space and the normal constant of P is k =1.

Theorem1.2.5:[1] Let (X , D) be a cone metric space, P be a normal cone with normal constant k. Let {x_n } be a sequence in X. Then:

(**i**) {x_n } converges to x if and only if D(x_n , x) → 0 as n → ∞.

(**ii**) If {x_n } converges to x and {x_n } converges to y, then x = y. That is the limit of {x_n } is unique.

(**iii**) If {x_n } converges to x, then {x_n } is a Cauchy sequence.

(**iv**) {x_n } is a Cauchy sequence if and only if D(x_n , x_m)→ 0 as $n,m \to \infty$.

(**v**)If {x_n } and {y_n } are two sequences in X and x_n→ x, y_n→ y (n→∞) Then D(x_n , y_n) →D(x , y) as (n → ∞).

Proof :

(i): Suppose that {x_n } converges to x. For every real ε > 0; choose c ∈ E with c ≫ 0 and k‖c‖ < ε.

Then, there is N such that for all n > N, D(x_n ,x) ≪ c. So that when n > N, ‖D(x_n , x)‖ ≤ k‖c‖ < ε. This means D(x_n , x) →0 as n→∞.

Conversely; suppose that $D(x_n, x) \to 0$ as $n \to \infty$ for $c \in E$ with $c \gg 0$ there is $\delta > 0$ such that $\|x\| < \delta$ implies that $c - x \in P^\circ$ for this δ there is N, such that for all $n > N$, $\|D(x_n, x)\| < \delta$, so $c - D(x_n, x) \in P^\circ$. This means $D(x_n, x) \ll c$. Therefore, $\{x_n\}$ converges to x.

(ii): For any $c \in E$ with $c \gg 0$, there is N such that for all $n > N$, $D(x_n, x) \ll c$ and $D(x_n, y) \ll c$, we have:

$D(x, y) \leq D(x_n, x) + D(x_n, y) \leq 2c$. Hence $\|D(x, y)\| \leq 2k\|c\|$ since c was arbitrary $\Rightarrow D(x, y) = 0$. Therefore, $x = y$.

(iii): For any $c \in E$ with $c \gg 0$, there is N such that for all $n, m > N$, $D(x_n, x) \ll \frac{c}{2}$ and $D(x_m, x) \ll \frac{c}{2}$. Hence; $D(x_n, x_m) \leq D(x_n, x) + D(x_m, x) \ll c$.

Therefore, $\{x_n\}$ is a Cauchy sequence.

(iv): Suppose that $\{x_n\}$ is a Cauchy sequence. For every $\varepsilon > 0$, choose $c \in E$ with $c \gg 0$ and $k\|c\| < \varepsilon$. Then there is N, for all $n, m > N$ $D(x_n, x_m) \ll c$. So that $\|D(x_n, x_m)\| \leq k\|c\| < \varepsilon$, when $n, m > N$. This means; $D(x_n, x_m) \to 0$ as $(n, m \to \infty)$.

Conversely; suppose that $D(x_n, x_m) \to 0$ $(n, m \to \infty)$ for $c \in E$ with $c \gg 0$, there is $\delta > 0$, such that $\|x\| < \delta$ implies $c - x \in P^\circ$ for this δ there is N, such that for all $n, m > N$, $\|D(x_n, x_m)\| < \delta$. So, $c - D(x_n, x_m) \in P^\circ$ thus, $D(x_n, x_m) \ll c$. Therefore $\{x_n\}$ is a Cauchy sequence.

(v): For every $\varepsilon > 0$, choose $c \in E$ with $c \gg 0$ and $\|c\| < \frac{\varepsilon}{4k+2}$.

From $x_n \to x$ and $y_n \to y$, we have:

$D(x_n, y_n) \leq D(x_n, x) + D(x, y) + D(y_n, y) \leq D(x, y) + 2c$

$D(x, y) \leq D(x_n, x) + D(x_n, y_n) + D(y_n, y) \leq D(x_n, y_n) + 2c$

Hence, $0 \leq D(x, y) + 2c - D(x_n, y_n) \leq 4c$.

Now,

$\|D(x_n , y_n)-D(x , y)\| \leq \|D(x , y)+2c-D(x_n , y_n)\|+\|2c\| \leq (4k+2)\|c\| < \varepsilon$

Therefore, $D(x_n , y_n) \rightarrow D(x , y)$ as $(n \rightarrow \infty)$.

Proposition 1.2.6 : [15]If $\{x_n\}$ is a decreasing sequence (via the partial ordering obtained by the cone P) such that $x_n \rightarrow u$ then, $u = \inf\{x_n : n \in N\}$.

Proof: Since $\{x_n\}$ is a decreasing sequence, $x_m - x_n \in P$ for all $n \geq m$ and $(x_m - x_n) \rightarrow (x_m - u)$, $\forall m$. Then closeness of P implies that $u \leq x_m$, $\forall m$. To see that u is the greatest lower bound of $\{x_n\}$, assume that $v \in E$ satisfies $x_m \geq v$, $\forall m$, from $(x_m - v) \rightarrow (u - v)$ and the closedness of P we get $u - v \in P^{\circ}$ or $v \leq u$ which shows that: $u = \inf\{x_n : n \in N\}$.

3. Cone Normed Spaces

Definition1.3.1: [16] Let X be a real vector space. Suppose that the mapping $\|\cdot\|_p : X \to E$ such that:

(i) $\|x\|_p \geq 0$, for all $x \in X$ and $\|x\|_p = 0$ if and only if $x = 0$.

(ii) $\|\alpha x\|_p = |\alpha|\|x\|_p$, for all $x \in X$ and $\alpha \in \mathbb{R}$.

(iii) $\|x+y\|_p \leq \|x\|_p+\|y\|_p$, for all $x , y \in X$.

Then $\|\cdot\|_p$ is called a cone norm on X and $(\|\cdot\|_p , X)$ is called a **cone normed space**.

It is easy to show that every normed space is a cone normed space by putting $E = \mathbb{R}$, $P = [0 , \infty)$.

Example1.3.2: [16] Let $E = \ell^1$, $P = \{\{x_n\} \in E : x_n \geq 0, \forall n\}$ and $(X , \|\cdot\|)$ be a normed space and $\|\cdot\|_p : X \to E$ defined as $\|x\|_p = \left\{\frac{\|x\|}{2^n}\right\}$. Then P is

a normal cone with k =1 and $(X , \|\cdot\|_p)$ is a cone normed space.

Remark1.3.3: [8] Let $(X, \|\cdot\|_p)$ be a cone normed space, set $D(x , y)= \|x - y\|_p$, it is easy to show that (X , D) is a cone metric space, D is called "the cone metric induced by the cone norm $\|\cdot\|_p$".

Proof: For all $x , y, z \in X$

1)$D(x , y)=0$ if and only if $\|x-y\|_p=0$ if and only if $x-y=0 \Leftrightarrow x = y$.

2)$D(x , y) = \|x - y\|_p = \|-(y-x)\|_p = \|y-x\|_p = D(y, x)$.

3)$D(x , y) =\|x - y\|_p =\|x-z+z-y\|_p \leq \|x-z\|_p + \|z-y\|_p = D(x , z) +D(z , y)$.

The following example is given to show that cone metrics do not necessarily produce cone norms.

Example1.3.4: [12] Let X= ℓ^1, P=[0,∞) and let E = ℝ. Let for x , y ∈ X d(x , y)= $\sum_{n=1}^{\infty}|x_n - y_n|$, then let D(x , y)= $\frac{d(x,y)}{1+d(x,y)}$.

It is easy to see that D is a cone metric relative to the cone P which is not compatible with any cone norm.

Remark1.3.5: [8] Convergence in cone normed space is described by the cone metric induced by the norm. For example, a sequence $x_n \in X$ is said to converge to an element x ∈ X, if for all c ≫0 there exist n∘ such that $D(x_n , x)=\|x_n - x\|_p \ll c$ for all n > n∘ . Hence, a sequence $x_n \to x$ if and only if $\|D(x_n , x)\| = \|\|x_n - x\|_p\| \to 0$ as n → ∞.

Definition1.3.6: [8] A sequence $x_n \in X$ is called **Cauchy sequence** if for all c≫ 0 there exists n∘ such that $D(x_n , x_m)=\|x_n - x_m\|_p \ll c$ for all n, m > n∘. Equivalently, if $\lim_{m,n\to\infty}\|D(x_n,x_m)\| = \lim_{m,n\to\infty}\left\|\|x_n - x_m\|_p\right\| = 0$.

Definition1.3.7: [8] We say that the cone normed space (X, $\|\cdot\|_p$) is a **cone Banach space** when the induced cone metric of $\|\cdot\|_p$ is complete.

Example1.3.8: [8] Let (E, $\|\cdot\|_p$), E=R^2, P={(x , y) : x ≥ 0 , y ≥ 0}. The function $\|\cdot\|_p$ defined by $\|(x , y)\|_p = (\alpha|x|, \beta|y|)$, α , β > 0 is a cone normed space and a cone Banach space.

Proof:

i)$\|(x , y)\|_p > 0$, $\|(x , y)\|_p = 0 \Leftrightarrow (\alpha|x| , \beta|y|)=(0 , 0) \Leftrightarrow \alpha|x|=0$ and $\beta|y|=0 \Leftrightarrow$ x = 0, y = 0 ⟺ (x , y) = (0 , 0).

ii)$\|a(x , y)\|_p=\|(ax , ay)\|_p=(|ax| , |ay|)=|a|(x , y)= |a|\|(x , y)\|_p$.

iii)$\|(x , y)+(z , w)\|_p = \|(x+z , y+w)\|_p=(|x+z| , |y+w|) \leq (|x|+|z| , |y|+|w|) = (|x| , |y|)+(|z| , |w|) = \|(x , y)\|_p + \|(z , w)\|_p$

Hence, $\|(x , y)+(z , w)\|_p \leq \|(x , y)\|_p+\|(z , w)\|_p$.

Therefore, (E , $\|\cdot\|_p$) is a cone normed space.

Now, let $z_n = (x_n , y_n) \in R^2$ be a Cauchy sequence, hence by (Definition.1.3.6)

$$\lim_{m,n\to\infty}\left\|\,.\|z_n - z_m\|_p\right\| = \lim_{m,n\to\infty}\left\|\,.\|x_n - x_m, y_n - y_m\|_p\right\| =$$

$$\lim_{m,n\to\infty}\left\|(\alpha|x_n - x_m|, \beta|y_n - y_m|)\right\| = \lim_{m,n\to\infty}\sqrt{\alpha^2|x_n - x_m|^2 + \beta^2|y_n - y_m|^2} = 0.$$

Therefore, $|x_n - x_m| \to 0$, $|y_n - y_m| \to 0$ as n , m → ∞. Hence, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in the field ℝ. One can find x , y ∈ℝ such that $|x_n - x\;| \to 0, |y_n - y\;| \to 0$ as n , m → ∞.

We shall show that $z_n = (x_n , y_n) \to z = (x , y)$ in cone norm space and hence (R^2, $\|\cdot\|_p$) is complete.

$$\lim_{n\to\infty}\left\|\,.\|z_n - z\|_p\right\| = \lim_{n\to\infty}\left\|\,.\|(x_n - x, y_n - y)\|_p\right\| = \lim_{n\to\infty}\left\|\left(\alpha|x_n - x|, \beta|y_n - y|\right)\right\| =$$

$$\lim_{n\to\infty}\sqrt{\alpha^2|x_n - x|^2 + \beta^2|y_n - y|^2} = 0.$$

Proposition1.3.9: [8] Every cone normed space is topological space.

Actually, the topology is given by:

T_P= { U ⊂ X : ∀x ∈ X , ∃ c ≫ 0 such that B(x , c) ⊂ U } where;

B(x , c) = { y ∈ X : $\|(x-y)\|_p \ll c$ }.

Theorem1.3.10: [8] The cone metric D induced by a cone norm on a cone normed space satisfies:(i)D(x + a , y + a) = D(x , y)

(ii)D(αx , αy) = |α|D(x , y)

Proof:

We have D(x + a , y + a) =$\|(x+a) - (y + a)\|_p$ =$\|(x-y)\|_p$= D(x , y)

and D(αx , αy) = $\|\alpha x - \alpha y\|_p$=$|\alpha|\|(x-y)\|_p$= |α|D(x , y).

If our cone is strongly minihedral, then we can define continuous functions.

Definition1.3.11: [8] A map T: (X , D) → (X , D) is called **continuous** at x ∈ X if for each V ∈T_P containing T(x), there exists U ∈T_P containing x such that T(U) ⊂V. If T is continuous at each x ∈ X, then it is called continuous.

Definition1.3.12: [16] Let (X , $\|\cdot\|_p$) be a cone normed space, a subset A ⊂ X is cone bounded if the set $\left\{\|x\|_p : x \in A\right\}$ has an upper bound.

In the following examples we see a linear mapping that is not cone bounded and a complete cone metric space which has no Cauchy sequences.

Example1.3.13: [16] Let E=R^2, P= {(x , y) ∈ E : x , y ≥ 0}, c_0 = (1 , 1) and let X be the set of all real–valued polynomials on the interval [0 , 1] and $\|\cdot\|_u$ is the supremum norm on X, that is :$\|f\|_u$ = sup{|f(x)|: x ∈ [0 , 1]} for all f ∈ X. Let $\|\cdot\|_p$: X → E be defined by $\|f\|_p = \|f\|_u c_0$, then (X , $\|\cdot\|_p$) is a cone normed space. Suppose D: X → X is defined by D(f) =f'. Then, D is a linear mapping that is not cone bounded.

Example1.3.14: [19]Let X = {a_1 , a_2 ,…} be a countable set of distinct points, E = (ℓ^2 , $\|\cdot\|_2$) and P=$\left\{\{x_n\}_{n\geq 1} \in \ell^2 : x_n \geq 0 (\forall n \geq 1)\right\}$. Put x_i=$\left\{\frac{3^i}{n}\right\}_{n\geq 1}$ for all i ≥ 1 and note that x_i ∈ ℓ^2 (i ≥1). Define the map D: X×X → P by D(x_i , x_j)={|x_i–x_j|}=$\left\{\frac{|3^i - 3^j|}{n}\right\}_{n\geq 1}$ .We see that (X , D) is a cone metric space, the normal constant of P is k =1 and there is no Cauchy sequence in (X , D). Hence (X , D) is a complete cone metric space.

Chapter Tow

Metrizability of cone metric spaces

1.**The generalization problem**

To generalize a mathematical concept has been a very long living problem. For example, the field of complex numbers is a generalization of the field of real numbers. Just to mention, the equation $x^2 +1 = 0$ has no real solution, but it has two complex solutions $\{ i , -i \}$. This generalization allowed us to extend very important functions from real to complex–valued functions.

Again, just to mention, the sine function is bounded on the real field but unbounded on the complex field, this occurs on the one hand. On the other hand, similar geometric feature may be deceptive. For instance, viewing the z-plane as being a copy of $\mathbb{R}^2$ may cause a feeling of full match of vector features of both fields. But just noting that the complex numbers make up a one–dimensional vector space while $\mathbb{R}^2$ is two–dimensional, will reviel the difference between the algebraic structure of the two spaces.

A very important concept in functional analysis is absorbency. For example, the Schatz's apple $S = \{(x , y) \in \mathbb{R}^2 : (x -1)^2 + y^2 =1\} \cup$

$\{(x , y)\in \mathbb{R}^2 : (x +1)^2 + y^2 =1\} \cup \{(x , y)\in \mathbb{R}^2 : x = 0 , -1\leq y \leq 1\}$

is absorbing if $\mathbb{R}^2$ is our underlying space but is not if complex field is our underlying space, see [27].

Our setting is no exception. We have an idea of generalization of the distance function on any set. The original classical setting is the non-negative value being assigned to the distance between two elements in a given set. Now, we would like to assign a member of a pre-assigned object in a Banach space. This object is assumed to obey some algebraic features.

We induce a mimic of the well-known order of $\mathbb{R}$. The definition of this order allows us to think in terms of the classical setting. In fact, every single word of the classical setting transforms (almost) automatically to the new setting. For example, convergence, generated topologies; the preservation of points under some recursively defined functions. Almost everything concerning limits transforms word for word here.

Having noted this, we made a deep study in the one direction of norms. Every classical theorem of limits gets back to where we started classically. In the literature, no new results were really different, with a few exceptions when your new object is not so nice in the new setting (not natural). The squeeze theorem may fail under the rigid setting. Also some tests to compare limits may fail. Other than that, everything you might think of makes a mimic in this new setting.

Remains to mention, our result was obtained in section 2 (**Our contributions**) of this chapter is to prove the metrizability of cone metric space. Our results are, in a way, to straighten the path of the renorming process. This work (ours) was accepted for publication in JNAA (see [20]).

It is worthwhile to mention, that the topic is still young, and many results of the authors of many articles may be easily extended or altered, or even proved to be untrue.

Right now, the work is concentrated on the improvement of a result on norms. The conjecture is: For any $M > 0$ there is a normal cone with constant of normality $k = M$. So far, the idea is to consider the subset of functions f defined on a subinterval of $[0 ,1]$, where $f(x)=ax- b$, with $a, b \geq 0$. It is believed (Dr. A. Hakawati believes so) to do the whole problem, probably in different spaces, (where $k = M$).

2. Historical introduction of metrizability of cone metric spaces

A metrizable space is a topological space that is homeomorphic to a metric space. That is, a topological space (X, T) is said to be metrizable if there is a metric $d : X\times X \to [0 , \infty)$ such that the topology induced by d is T.

One of the first widely-recognized metrization theorems was Urysohn's metrization theorem, it states that every Hausdorff second-countable regular space is metrizable.

Several other metrization theorems follow as simple corollaries to Urysohn's theorem. For example, a compact Hausdorff space is metrizable if and only if it is second-countable.

In the previous chapter we pointed out that any cone normed space is a topological space so it remains only the equivalent metric d.

Note: Every cone metric space is indeed metrizable.

Many authors showed that the cone metric spaces are metrizable and defined the equivalent metric using different approaches. In this chapter we will review some of those approaches and finally give our contribution in this direction.

Firstly, M.Khani and M.Pourmadian prove the metrizability of cone metric space in [14].The following theorem defines a metric d representing a cone metric space.

Theorem 2.2.1: [14] Let (X, E, P, D) be a cone metric space, α ∈ P° and c < 1 be in $\mathbb{R}^+$. Then there exists a metric d: X×X → $\mathbb{R}^+$ which induces the same topology on X as the cone metric topology induced by D. Moreover, a sequence $\{x_n\}$ is Cauchy in (X, E, P, D) if and only if it is Cauchy in (X, d). In particular, (X, E, P, D) is complete if and only if (X, D) is complete.

Their definition was:

$$\Lambda(x , y)=\begin{cases} D^{\min\{k : d(x,y) \ll d^k \alpha\}}, D(x,y) \neq 0 \\ 0, D(x,y) = 0 \end{cases}$$

and d(x , y)= $\inf\left\{\sum_{i=1}^{n-1} \Lambda(x_i, x_{i+1}) : x_1 = x, ..., x_n = y\right\}$

Despite the intricacies of their definition, cone metric spaces can in part be dealt with as the familiar metric spaces.

In [12], A.A. Hakawati and S. Al-Dwaik tried to answer the question of metrizability in the sense of best approximation. For the classical setting of best approximation theory readers can consult [3].

Also M. Asadi, S.M.Vaezpour and H.Soleimani in [6] tried to prove the metrizability by answering "in the negative" the important question " Are cone metric space a real generalization of metric space ? " by proving that every cone space is metrizable and the equivalent metric satisfies the same contractive conditions as the cone metric.

Their definition was $d(x , y)=\inf\{\|u\| : D(x , y) \leq u\}$; where $D: X\times X\rightarrow E$ any cone metric and $d: X\times X\rightarrow \mathbb{R}^+$ the equivalent metric to D.

Moreover, the same authors in[10] prove the metrizability of cone metric spaces by renorming the Banach space, then every cone P can be converted to a normal cone with constant k = 1.Their renorming was the following norm. $|\|\cdot\|| : E \rightarrow [0 , \infty)$ defined as:

$|\|x\||=\inf\{\|u\| : x \leq u\}+\inf\{\|v\| : v \leq x\}+\|x\|$, for all $x \in E$.

So every cone metric $D : X\times X \rightarrow E$ is equivalent to the metric defined by $d(x , y) = |\|D(x , y)\||$. Therefore, every cone metric defined on a Banach space is really equivalent to a metric.

It is important to know that this result of M. Asadi et al. in [10] has been disproved by Z. Kadelburg and S. Radenovich in [18] because their result cannot be true (an error appears in the last step of the proof, when proving that P is monotone (normal cone with normal constant k=1) in the new norm) this would imply that all cones in Banach spaces are normal, which is obviously not true.

Also, some other authors in [11] showed that the result of M. Asadi et al. does not hold. Moreover, they titled their works by " On non metrizability of cone metric spaces ". The authors their presented the next counter example which shows that the main theorem in [10] does not hold.

Example 2.2.2 :[11] Let E = $C_R^2([0,1])$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$ and consider the cone P ={ f ∈ E : f ≥ 0 } then P is a non-normal cone let f(x) = x and g(x) = x^2 , ∀ x ∈ [0 , 1].

Then, clearly, 0 ≤ g ≤ f. Further $\|f\| = \|f\|_\infty + \|f'\|_\infty$ = 1+1 = 2 and $\|g\| = \|g\|_\infty + \|g'\|_\infty$ = 1+2 = 3.

Since f ∈ P,

$|\|f\|| = \inf\{\|u\| : f \leq u\} + \inf\{\|v\| : v \leq f\} + \|f\|$

$= \inf\{\|u\| : f \leq u\} + 0 + \|f\|$

$= \|f\|_\infty + 2 = 1 + 2 = 3$

Also, g ∈P, $|\|g\|| = \inf\{\|u\| : g \leq u\} + \inf\{\|v\| : v \leq g\} + \|g\|$

$= \inf\{\|u\| : g \leq u\} + 0 + \|g\|$

$= \|g\|_\infty + 3 = 1 + 3 = 4$

According to M. Asadi et al. in [10], P is a normal cone with normal constant K =1. Hence, 0 ≤ g ≤ f implies that $|\|g\|| \leq |\|f\|| \Rightarrow 4 \leq 3$ which is a contradiction.

3. Our contributions

Our main result states that we can convert every strongly minihedral normal cone to a normal cone with K =1 by giving a new norm to our Banach space and consequently due to this approach every cone metric space is really a metric space and every theorem in metric space is valid for cone metric space automatically.

First of all, we need the following lemmas.

Lemma 2.3.1 :[20] Suppose P is a strongly minihedral cone in a real Banach space E. Then; for x , y ∈E; we have:

(1) $\inf\{w : w \geq x + y\} = \inf\{u : u \geq x\} + \inf\{v : v \geq y\}$

(2) $\sup\{w : w \leq x + y\} = \sup\{u : u \leq x\} + \sup\{v : v \leq y\}$

Proof:

(1)Let $U=\{w : w \geq x+y\}$, $U_1=\{u : u \geq x\}$, $V_1=\{v : v \geq y\}$

Let e, e_1 and e_2 be inf(U), inf(U_1) and inf(V_1), respectively.

Now, if $u \in U$ then $u \geq x+y$, so $u-y \geq x$ and so $u-y \in U_1$

So $e_1 \leq u-y \Longrightarrow u-e_1 \geq y$ and so $u - e_1 \in V_1$.

So $e_2 \leq u- e_1$ and so $e_1 + e_2 \leq u$ (this is true $\forall\, u \in U$). So

(1) $$e_1 + e_2 \leq e$$

For the other inequality, note that by the very definition of e_1 and e_2 we have: $e_1 + e_2 \geq x+y$ so $e_1 + e_2 \in U$, and hence

(2) $$e \leq e_1 + e_2$$

by (1) and (2) we conclude that : $e = e_1 + e_2$

i.e. $\inf\{w : w \geq x + y\} = \inf\{u : u \geq x \}+ \inf\{v : v \geq y\}$

Likewise, $\sup\{w : w \leq x + y\} = \sup\{u : u \leq x\} + \sup\{v : v \leq y\}$. ■

Lemma 2.3.2: [20] For $0 \leq x \leq y$ we have:

(1)$\|\sup\{u : u \leq x\}\| \leq \|\sup\{u' : u' \leq y\}\|$

(2)$\|\inf\{v : x \leq v\}\| \leq \|\inf\{v' : y \leq v'\}\|$

Proof : For (1) let $U = \{u : u \leq x\}$, $U' = \{u' : u' \leq y\}$

Since $0 \leq x \leq y$ we have $x \in U' \Longrightarrow x \leq \sup(U')$ so $\forall\ u \in U$ we have

$u \leq \sup(U')$ so $\sup(U) \leq \sup(U')$

Hence, $\|\sup\{u : u \leq x\}\| \leq \|\sup\{u' : u' \leq y\}\|$

Similarly, one can show: $\|\inf\{v : x \leq v\}\| \leq \|\inf\{v' : y \leq v'\}\|$. ■

Theorem 2.3.3: [20] Let (E ,$\|\cdot\|$) be a real Banach space with a strongly minihedral normal cone P. Then there exists a norm $[\cdot]$on E with respect to which P is a normal cone with normal constant K =1 .

Proof : Define $[\cdot]: E \to [0 , \infty)$ by:

$[x]=\|\inf\{u : x \leq u\}\|+\|\sup\{v : v \leq x\}\|$, for all $x \in E$

It is clear that, if $x = 0$ then $[x] = 0$, for all $x \in E$

If $[x] = 0$ then $\exists\ u_n$, $v_n \in E$ such that: $v_n \leq x \leq u_n$ where $u_n \to 0$, $v_n \to 0$ as $n \to \infty$. Since P is a normal cone then we get $x = 0$.

Therefore, $[x] = 0$ if and only if $x = 0$.

Now, $[-x] = \|\inf\{u : -x \leq u\}\| + \|\sup\{v : v \leq -x\}\|$

$= \|-\sup\{-u : -x \leq u\}\| + \|-\inf\{-v : v \leq -x\}\|$

$= \|\sup\{-u : -u \leq x\}\| + \|\inf\{-v : x \leq -v\}\|$

$$= \|\sup\{u : u \leq x\}\| + \|\inf\{v : x \leq v\}\| = [x]$$

For $\lambda > 0$,

$$[\lambda x] = \|\inf\{u : \lambda x \leq u\}\| + \|\sup\{v : v \leq \lambda x\}\|$$

$$= \|\inf\{\lambda (\frac{1}{\lambda} u) : x \leq \frac{1}{\lambda} u\}\| + \|\sup\{\lambda (\frac{1}{\lambda} v) : \frac{1}{\lambda} v \leq x\}\|$$

$$= \lambda\|\inf\{\frac{1}{\lambda} u : x \leq \frac{1}{\lambda} u\}\| + \lambda\|\sup\{\frac{1}{\lambda} v : \frac{1}{\lambda} v \leq x\}\| = \lambda[x]$$

Therefore, $[\lambda x] = |\lambda|[x]$, $\forall\, x \in E$ and $\lambda \in \mathbb{R}$.

Now, we prove the triangle inequality, using lemma2.3.1

$$[x + y] = \|\inf\{u : x + y \leq u\}\| + \|\sup\{v : v \leq x + y\}\|$$

$$= \|\inf\{u : x \leq u\} + \inf\{u : u \leq y\}\| + \|\sup\{v : v \leq x\} + \sup\{v : v \leq y\}\|$$

$$\leq \|\inf\{u : x \leq u\}\| + \|\inf\{u : u \leq y\}\| + \|\sup\{v : v \leq x\}\| + \|\sup\{v : v \leq y\}\|$$

$$= \|\inf\{u : x \leq u\}\| + \|\sup\{v : v \leq x\}\| + \|\inf\{u : u \leq y\}\| + \|\sup\{v : v \leq y\}\|$$

$$= [x] + [y].$$

Therefore, $[x+y] \leq [x] + [y]$ and hence, $[\cdot]$ is a norm on E.

Finally, with respect to this norm $[\cdot]$, by lemma2.3.2, P is a normal cone with normal constant $k = 1$. ■

So our belief is that a cone metric space is really a metric space and every theorem in metric spaces is valid for cone metric space automatically. This enforces the feeling that cone metric spaces are not real generalization of metric spaces.

Chapter Three

Topological cone metric spaces

The generalizing problem occupies a large area of the interest of mathematicians. On the one hand, it certainly opens wide scopes of applications, and on the other, it helps find quicker solutions to problems with long and perhaps difficult proofs. There are many occasions one can mention. Just for example, the uniform boundedness principle made short, the long proof of theorems on the continuity (boundedness) of maps between Banach spaces. Some matrix maps between sequence spaces, we proved to be bounded using only a paragraph full statement. We refer the reader to any standard book on functional analysis which contains the uniform boundedness principle. Albert Wilansky proved this in his book: Modern Methods of Topological Vector Spaces, 1978.

In literature, we didn't find any specific problem our topic is trying to solve. This, probably, why things are only aimed to make things general. We were aware of this all the time. You cannot go for nowhere, but you can say that this way, or that, does not, or does, give you new fruitful results. It looks like that all authors do the same as we do.

Recently, in the 5^{th} Palestinian conference on modern trends in mathematics and physics, A. Hakawati suggested the following for a statement to the generalization problem:

The map $T: X \rightarrow X$ has a fixed point if and only if it satisfies some contraction condition with respect to some cone P.

One thing we would like to reemphasize is that, under the normality assumptions, there does not happen any intrinsic difference with the new type of measure.

In this chapter, topological questions were answered in cone metric spaces, where it was proved that cone metric spaces are: topological space, first countable space, Hausdorff space and T_4 space. And some other topological concepts and definition are generalized to cone metric spaces.

First of all, we need the following important lemmas:

Lemma 3.1.1: [5] Let (X, D) be a cone metric space. Then for all $c \gg 0$, $c \in E$, there is $\delta > 0$ such that $(c-x) \in P^{\circ}$ (i.e. $x \ll c$) whenever $\|x\| < \delta$, $x \in E$.

Proof: Since $c \gg 0$ and $c \in P^{\circ}$. Then, we can find $\delta > 0$ such that:

$\{x \in E : \|x - c\| < \delta \} \subset P^{\circ}$.

Now, if $\|x\| < \delta$ then $\|(c-x) - c\| = \|-x\| = \|x\| < \delta$, and then $(c-x) \in P^{\circ}$.

Lemma 3.1.2: [5] Let (X , D) be a cone metric space. Then, for all $c_1 \gg 0$ and $c_2 \gg 0$, $c_1, c_2 \in E$, there is $c \gg 0$, $c \in E$ such that $c \ll c_1$ and $c \ll c_2$.

Proof: Since $c_2 \gg 0$, then by Lemma3.1.1, we can find $\delta > 0$ such that $\|x\| < \delta$ this implies that $x \ll c_2$. Choose n_0 such that $\frac{1}{n_0} < \frac{\delta}{\|c_1\|}$. Take $c = \frac{c_1}{n_0}$. Then $\|c\| = \left\|\frac{c_1}{n_0}\right\| = \frac{\|c_1\|}{n_0} < \delta$ and therefore , $c \ll c_2$. But also it is clear that $c \gg 0$ and $c \ll c_1$.

Theorem 3.1.3: [5] Every cone metric space (X , D) is a topological space.

Proof: For all $c \gg 0$, $c \in E$, let $B(x , c)=\{y \in X : D(x ,y) \ll c\}$ and $\beta=\{ B(x , c) : x \in X ,\ c \gg 0\}$ so $T_c =\{U \subset X : \forall x \in U , \exists B \in \beta , x \in B \subset U\}$ is a topology on X. In fact, we have:

(T_1) $\emptyset, X \in T_c$.

(T_2) Let $U, V \in T_c$ and let $x \in U \cap V$. Then, $x \in U$ and $x \in V$, find $c_1 \gg 0$, $c_2 \gg 0$ such that $x \in B(x , c_1) \subset U$ and $x \in B(x , c_2) \subset V$, by lemma3.1.2 find $c \gg 0$ such that $c \ll c_1$ and $c \ll c_2$ .Then, clearly $x \in B(x , c) \subset B(x , c_1) \cap B(x , c_2) \subset U \cap V$. Hence, $U \cap V \in T_c$.

(T_3) Let $U_\alpha \in T_c$ for each $\alpha \in \Gamma$, and let $x \in \bigcup_{\alpha\in\Gamma} U_\alpha$ .Then $\exists \alpha_\circ \in \Gamma$ such that $x \in U_{\alpha_\circ}$. Hence, find $c \gg 0$ such that $x \in B(x , c) \subset U_\alpha \subset \bigcup_{\alpha\in\Gamma} U_\alpha$
That is, $\bigcup_{\alpha\in\Gamma} U_\alpha \in T_c$.

Theorem 3.1.4: Every cone metric space (X , D) is a Hausdorff space.

Proof: Let $x, y \in X$ such that ($x \neq y$) are two points in X, then $D(x , y) = c > 0$, so that $[B(x ,\frac{c}{2}) \cap B(y ,\frac{c}{2})] \in T_c$ and $B(x , \frac{c}{2}) \cap B(y ,\frac{c}{2}) = \emptyset$.

Definition 3.1.5: [8] Let $U \neq\emptyset$ and $V \neq\emptyset$ be two subsets of a cone metric space (X , D) Then the **distance** between U and V, denoted by $d(U ,V)$ is defined by $d(U ,V)=\inf\{D(u , v): u \in U , v \in V\}$. If $U=\{u\}$, we write $d(u , V)$ for $d(U , V)$.

Example 3.1.6: [8] Let $E = R^2$ and $P = \{(u, v): u \geq 0, v \geq 0\}$. We define $D: R^2\times R^2 \to E$ by $D((u_1 , u_2),(v_1 , v_2))= (|u_1 - v_1| , | u_2 - v_2|)$ and let

$U=\{(u, v) \in R^2 : 1 \leq u \leq 2, 3 \leq v \leq 4\}$, $V=\{(u, v) \in R^2 : 4 \leq u \leq 5, 1 \leq v \leq 4\}$. Then $d(U, V)= (2, 0)$

Definition 3.1.7: [8] Let (X , D) be a cone metric space. Then $U \subset X$ is called **bounded above** if there is $c \in E$, $c \gg 0$ such that $D(x, y) \leq c$ for all $x, y \in U$, and is called **bounded** if $\delta(U) = \sup\{D(x, y): x, y \in U\}$ exists in E. If the supremum does not exist, we say that U is unbounded.

Theorem 3.1.8: [8] Every cone metric space (X , D) is a first countable.

Proof: Let $q \in X$ and fix $c \gg 0$ where $c \in E$. We see that; $\beta_q=\{B(q, \frac{c}{n}) : n \in \mathbb{N}\}$ is a local base at q. Let U be open with $q \in U$.

Find $c_1 \gg 0$ such that $q \in B(q, c_1) \subset U$, also by lemma3.1.1, find $n_\circ$ such that that $\frac{c}{n_0} \ll c_1$. Therefore, $B(q, \frac{c}{n_0}) \subset B(q, c_1) \subset U$.

Definition 3.1.9: [8] Let (X , D) be a cone metric space. A subset $U \subset X$ is called **sequentially closed** if whenever $x_n \in U$ with $x_n \to x$ then $x \in U$.

Theorem 3.1.10: [8] Let (X , D) be a cone metric space. Then the ball $\overline{B(x, c)} = \{y \in X : D(x, y) \leq c\}$, $c \gg 0$, $c \in E$ is a sequentially closed.

Proof: Let $y_n \in \overline{B(x, c)}$ be a sequence such that $y_n \to y$. Then $D(y_n, x) \leq c$ and $D(y_n, x) \to 0$ as $n \to \infty$. Then $y \in \overline{B(x, c)}$ if and only if $D(x, y) \leq c$ if and only if $c - D(x, y) \in P$. But then $D(y_n, x) \to D(x, y)$, since P is closed, then $\lim_{n\to\infty}(c - D(y_n, x)) = c - D(x, y) \in P$.

Lemma 3.1.11: [8] Let P be a normal cone in E and $\{x_n\}$, $\{y_n\}$ be two sequences in E. If $y_n \to y$, $x_n \to x$ as $n \to \infty$ in $(E, \|\cdot\|)$ and $x_n \leq y_n$ for all n, then $x \leq y$.

Proof: $x_n \leq y_n$ implies that $(y_n - x_n) \in P$. Since P is closed and $(y_n - x_n) \to (y - x)$ then $(y - x) \in P$. Hence, $x \leq y$.

Theorem 3.1.12: [8]Let (X , D) be a cone metric space and $U \subset X$ where $U \neq \emptyset$.Then $x \in \overline{U}$ if and only if D(x , U) = 0.

Proof : Suppose $x \in \overline{U}$. Then, for $c \gg 0$ and each $n \in \mathbb{N}$, $B(x , \frac{c}{n}) \cap U \neq \emptyset$. Hence, for each $n \in \mathbb{N}$ there is $u_n \in U$ such that: $0 \leq D(x , U) \leq D(x , u_n) < \frac{c}{n}$. Hence, $\|D(x , U)\| \leq k\frac{\|c\|}{n}$, for all $n \in \mathbb{N}$. Therefore, D(x , U) = 0.

Conversely, let $V \in T_c$ be open in (X , D) such that $x \in V$, then we find $c \gg 0$ such that $B(x , c) \subset V$. But, since $0 = D(x , U) < c$, find $u \in U$ such that $D(x , u) < c$. That is $u \in U \cap B(x , c) \subset U \cap V$.

Definition 3.1.13 :[4] A topological space X is a **normal space** if, given any disjoint closed sets F and G, there are open neighborhoods U of F and V of G that are also disjoint. More intuitively, this condition says that F and G can be neighborhoods.

Definition 3.1.14: [4] A **T_4 space** is a T_1 space X that is normal; this is equivalent to X being normal and Hausdorff.

Theorem 3.1.15: [4] Every cone metric space (X , D) is a T_4 space.

Proof: As already mentioned (X , D) is a Hausdorff space. To show that X is normal, let F and G be two closed disjoint subsets of X and define $U=\{x \in X : D(x , F) < D(x , G)\}$ and $V=\{x \in X: D(x , F)>D(x , G)\}$ from the definition of U and V we see $U \cap V = \emptyset$. Furthermore, if $a \in F$, then $D(a , F) = 0$, $a \notin G$ and, since G is closed, $D(a , G) > 0$. According to (theorem 3.1.11), $D(a , F) < D(a , G)$ so that $a \in U$, it follows that $F \subset U$. Similarly, $G \subset V$.

Now, if we show that U and V are open then we will be done. To show that U is open, let $x_\circ \in U$ then $c_1 = D(x_\circ, F) < D(x_\circ, G) = c_2$. Since $c_2-c_1> 0$ [i.e. $c_2-c_1 \in P$, $c_2 \neq c_1$], we may define $c=\frac{1}{2}(c_2-c_1)$ and consider the basic open $B(x_\circ , \frac{c}{2})$. Let $x \in B(x_\circ , \frac{c}{2})$, then for each $s \gg 0$ and by the definition of $D(x_\circ, F)$, there exists $a \in F$ such that $D(x_\circ, a)< c_1+s$. Therefore, $D(x, F) \leq D(x, a) \leq D(x, x_\circ)+D(x_\circ, a) < \frac{c}{2}+ c_1+s$. Then, it follows that $D(x, F) \leq \frac{c}{2}+ c_1 =\frac{1}{4}(3c_1+c_2)$. Also, for $b \in G$, we have $D(b, x_\circ) \leq D(b, x)+D(x, x_\circ)$ and since $D(x_\circ, G) \leq D(x_\circ, b)$ and $D(x, x_\circ) \leq \frac{c}{2}$. We may write $D(b, x) +\frac{c}{2} > D(x_\circ, G) = c_2$. Thus, $D(b, x) > c_2-\frac{c}{2}=\frac{1}{4}(3c_2+c_1)$.Then, by noting that $c_2+3c_1 < 3c_2+c_1$ we conclude that $D(x , F) < D(x , G)$. That is, $x \in U$ and hence U is open subset of X. The same reasoning shows V is also open subset of X.

Definition 3.1.16: [8] A map $T :(X , D) \rightarrow (X , D)$ is called **continuous** at $x \in X$ if for each $V \in T_c$ containing $T(x)$ there exists $U \in T_c$ containing x such that $T(U) \subset V$. If T is continuous at each $x \in X$, then it is called continuous.

Definition 3.1.17: [8] Let (X, D) be a cone metric space. A map $T:(X, D) \to (X, D)$ is called sequentially **continuous** if for $x_n \in X$ such that $x_n \to x$ implies $T(x_n) \to T(x)$.

Theorem 3.1.18: [8] Let (X , D) be a cone metric space, and assume that $T: (X , D) \to (X , D)$ be a map. Then, T is continuous if and only if T is sequentially continuous.

Proof: Assume $x_n \to x$ and let $c \gg 0$ since T is continuous at $x \in X$, then find $c_1 \gg 0$ such that $T(B(x , c_1)) \subset (B(T(x) ,c))$. By convergence of x_n, find $n_\circ$ such that $D(x_n , x) \ll c_1$, $\forall\ n \geq n_\circ$. But then, $D(T(x_n),T(x)) \ll c$ $\forall\ n \geq n_\circ$. Since (X , D) is a first countable topological space, then the converse holds.

Definition 3.1.19: [8] Let (X , D) be a cone metric space and $c \gg 0$, $c \in E$. A finite subset $N= \{c_1, c_2,.., c_n\}$ of X is called a **c-net** for the subset $U \subset X$ if for each $p \in U$ there is $c_{i\circ} \in N$ such that $D(p , c_{i\circ}) \ll c$.

Definition 3.1.20: [8] Let (X , D) be a cone metric space. A subset U of (X, D)is called **totally bounded** if for each $c \gg 0$, $c \in E$, U can be composed into a finite union of sets N_i , i= 1 , 2 ,…. , n (i.e. $U \subset \bigcup_{i=1}^{n} N_i$) where $\delta(N_i) \ll c$.

Theorem 3.1.21: [8] Let (X , D) be a cone metric space, and $U \subset X$. Then U is totally bounded if and only if for each $c \gg 0$, $c \in E$, U has a c-net.

Proof: Assume U is totally bounded and let $c \gg 0$, $c \in E$. Then find $N_1, N_2, \ldots, N_n$ such that $U \subset \bigcup_{i=1}^{n} N_i$, $\delta(N_i) \ll c$.

From each N_i choose an element c_i, $i = 1, 2, \ldots, n$. Let $N=\{c_1, c_2, \ldots, c_n\}$. We show that N is a c-net for U.

Let $p \in U$. Then there exists $c_{i_\circ}$, $i_\circ = \{1, 2, \ldots, n\}$ such that $p \in N_{i_\circ}$. Using that both p and $c_{i_\circ}$ are in $N_{i_\circ}$ and that $\delta(N_i) \ll c$, we conclude that $D(p, c_{i_\circ}) \ll c$.

Conversely, let $c \gg 0$, $c \in E$. Then find a finite set $N=\{c_1, c_2, \ldots, c_n\}$ such that for each $p \in U$ there is $c_{i_\circ} \in N$ with $D(p, c_{i_\circ}) \ll c$. Let $N_i = B(c_i, c)=\{x \in X : D(x, c_i) \ll c\}$, $i=1, 2, \ldots, n$. Then clearly $U \subset \bigcup_{i=1}^{n} N_i$ and $\delta(N_i) \ll c$.

Definition 3.1.22: [8] Let (X , D) be a cone metric space. An element $c \gg 0$, $c \in E$, is called a **Lebesgue element** of a cover $\mathcal{C} = \{G_i\} \subset T_c$, $i= 1, 2, 3, \ldots$ for a subset U of (X , D) if for each subset V of U with $\delta(V) < c$ there exists $G_{i_\circ} \in \mathcal{C}$ such that $V \subset G_{i_\circ}$.

Definition 3.1.23: [8] Let (X , D) be a cone metric space. A subset U of (X , D) is called **compact** if each cover of U by subsets from T_c contains a finite subcollection that also covers U.

Definition 3.1.24: [8] Let (X , D) be a cone metric space and $U \subset X$. If for any sequence $\{x_n\}$ in U, there exists a subsequence $\{x_{ni}\}$ of $\{x_n\}$ such that $\{x_{ni}\}$ is convergent in U, then U is called **sequentially compact**.

Theorem 3.1.25: [8] Let (X , D) be a cone metric space and $U \subset X$. If U is sequentially compact, then it is totally bounded.

Proof: Assume there exists $c \gg 0$ where $c \in E$ such that U cannot have a c-net. Hence, for fixed $x_1 \in U$ there exists $x_2 \in U$ such that: $c-D(x_1 , x_2) \notin P^{\circ}$, then also $\{x_1 , x_2\}$ cannot be a c-net for U, hence there is $x_3 \in U$ such that $c- D(x_1 , x_3) \notin P^{\circ}$ and $c-D(x_3 , x_2) \notin P^{\circ}$. Like this method we construct a sequence $x_n \in U$ such that $c-D(x_n, x_m) \notin P^{\circ}$, $\forall$ n, m $\in \mathbb{N}$. So, any subsequence of $\{x_n\}$ cannot be Cauchy and $\{x_n\}$ cannot have convergent subsequence. Therefore, U is not sequentially compact.

Theorem 3.1.26: [8] Let (X , D) be a cone metric space. Then, a subset $U \subset X$ is compact if and only if U is sequentially compact.

Proof: Let $\mathcal{C} = \{G_i\}_{i \in I}$ be an open cover for U. Since U is sequentially compact, there is $c \gg 0$ where $c \in E$ such that for any subset $V \subset U$ with $\delta(V) < c$, there is $i_{\circ} \in I$ with $V \subset G_{i_{\circ}}$. Since U is totally bounded then $U \subset \bigcup_{i=1}^{n} N_i$ where $\delta(N_i) \ll c$. Hence, for each i=1 , 2 ,..., n find $G_1 , G_2 , \dots ,$ $G_n \in \mathcal{C}$ such that $N_i \subset G_i$. That is, $U \subset \bigcup_{i=1}^{n} N_i \subset \bigcup_{i=1}^{n} G_i$ and hence U is compact.

However, it is noted that, since every cone metric space is a topological space, then compact CMSs are sequentially compact.

Chapter Four

Fixed point theory in cone metric spaces

1. Fixed point theorems of contractive mappings in cone metric spaces

Fixed point theory occupies a prominent place in the study of metric spaces. One of the important questions that may arise in this connection is "whether metric spaces really provide enough space for this theory or not?"

Recently, Huang and Zhang, in [1], rather implied that the answer is no. Actually, they were who introduced the nation of cone metric space. Also, they studied the existence and uniqueness of the fixed point for a self-map T on a cone metric space (X , D).

The authors in [1] considered different contractive conditions on T. They also assumed (X , D) to be complete when P is a normal cone, and (X , D) to be sequentially compact when P is a regular cone.

Later, in [2], Rezapour and Halmbarani improved some of the results in [1] by omitting the normality assumption of the cone P, when (X, D) is complete, which is a milestone in developing fixed point theory in cone metric spaces.

Moreover, in [17], H.P.Masiha, F.Sabetghadam and A.H.Sanatpour focused on the regularity condition of the cone P, when (X , D) is sequentially compact, and they improved the basic theorem in [1] (theorem 2.2) by omitting the regularity assumption and considered the weaker condition of normality on the cone P.

In this chapter we will study some fixed point theorems in cone metric space.

Definition 4.1.1: A contraction mapping on a cone metric space (X, D) is a function T from X to itself with the property that there is some non-negative real number K ∈ [0 ,1) such that: D(T(x), T(y)) ≤ KD(x , y) for all x , y ∈ X.

The smallest such value of K is called the Lipschitz constant of T and contractive maps are sometimes called Lipschitzian maps.

Remark 4.1.2: If the contraction condition is instead satisfied for all K ≤ 1 then the mapping is said to be a non-expansive map.

Example 4.1.3: [21] Let X = [0, 1] ∪ [2, ∞[and D : X× X → [0 , ∞) defined by D(x , y) = |x −y |. Define f : X→ X by: $f(x)=\begin{cases} \frac{1}{2}+\frac{1}{2}x & \text{If } x \in [0, 1] \\ 1+\frac{1}{2}x & \text{If } x \in [2, \infty[\end{cases}$

Then, (X , D) is a complete cone metric space and f is a non-expansive mapping.

A contraction mapping has at most one fixed point. Moreover, the Banach fixed point theorem states that every contraction mapping on a nonempty complete metric space has a unique fixed point. And for any x in X, the iteration function sequence: x, T(x), T(T(x)), …. converges to the fixed point.

Theorem 4.1.4: [1] Let (X , D) be a complete cone metric space, P be a normal cone with normal constant k. Suppose the mapping T: X→X satisfies the contractive condition: $D(T(x), T(y)) \leq K\, D(x, y)$ for all $x, y \in X$, where $K \in [0, 1)$ is a constant. Then, T has a unique fixed point in X, and for each $x \in X$, the iterative sequence $\{T^n(x)\}_{n\geq 1}$ converges to the fixed point.

Proof: choose $x_\circ \in X$ set $x_1 = T(x_\circ)$, $x_2 = T(x_1) = T^2(x_\circ)$, $x_3 = T(x_2) = T^3(x_\circ)$, ... , $x_{n+1} = T(x_n) = T^{n+1}(x_\circ)$

We have:

$$D(x_{n+1}, x_n) = D(Tx_n, Tx_{n-1}) \leq K\, D(x_n, x_{n-1})$$

$$\leq K^2 D(x_{n-1}, x_{n-2}) \leq \ldots \leq K^n D(x_1, x_\circ)$$

So for $n > m$;

$$D(x_n, x_m) \leq D(x_n, x_{n-1}) + D(x_{n-1}, x_{n-2}) + \ldots + D(x_{m+1}, x_m)$$

$$\leq (K^{n-1} + K^{n-2} + \ldots K^m) D(x_1, x_\circ) \leq \frac{K^m}{1-K} D(x_1, x_\circ)$$

By normality assumption of P, we get: $\|D(x_n, x_m)\| \leq \frac{K^m}{1-K} k \|D(x_1, x_\circ)\|$

this implies $D(x_n, x_m) \to 0$ (as $n, m \to \infty$), hence $\{x_n\}$ is a Cauchy sequence. So, by the completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

Now; $D(Tx^*, x^*) \leq D(Tx_n, Tx^*) + D(Tx_n, x^*) \leq K\, D(x_n, x^*) + D(x_{n+1}, x^*)$

So; $\|D(Tx^*, x^*)\| \leq k(K \| D(x_n, x^*)\| + \|D(x_{n+1}, x^*)\|) \to 0$

Hence; $\|D(Tx^*, x^*)\| = 0$ and so $D(Tx^*, x^*) = 0$

Therefore, $Tx^* = x^*$, so x^* is a fixed point of T.

Now, to see the uniqueness, let y^* be another fixed point of T then: $D(x^*, y^*) = D(Tx^*, Ty^*) \leq KD(x^*, y^*)$.
Hence, $\|D(x^*, y^*)\| = 0$ and so $x^* = y^*$.

Theorem 4.1.5: [1] Let (X , D) be a sequentially compact cone metric space, P be a regular cone. Suppose the mapping $T: X \rightarrow X$ satisfies the contractive condition: $D(Tx, Ty) < D(x, y)$ for all $x, y \in X$, $x \neq y$. Then T has a unique fixed point in X.

Proof: choose $x_\circ \in X$ set $x_1 = T(x_\circ)$

$$x_2 = T(x_1) = T^2(x_\circ)$$

$$x_3 = T(x_2) = T^3(x_\circ), \ldots, x_{n+1} = T(x_n) = T^{n+1}(x_\circ)$$

If for some n, $x_{n+1} = x_n$ then x_n is a fixed point on T, the proof is complete.
So we assume that for all n, $x_{n+1} \neq x_n$ set $d_n = d(x_n, x_{n+1})$, Then: $d_{n+1} = D(x_{n+1}, x_{n+2}) = D(Tx_n, T_{x+1}) < D(x_n, x_{n+1}) = d_n$ Therefore, d_n is decreasing sequence bounded below by 0. Since P is regular, there is $d^* \in E$ such that $d_n \rightarrow d^*$ as $n \rightarrow \infty$. So from the sequentially compactness of X, there are subsequence $\{x_{ni}\}$ of $\{x_n\}$ and $x^* \in X$ such that $x_{ni} \rightarrow x^*$ as $i \rightarrow \infty$.
We have $D(Tx_{ni}, Tx^*) \leq D(x_{ni}, x^*)$, $i = 1, 2, \ldots$
So: $\|D(Tx_{ni}, Tx^*)\| \leq k\|D(Tx_{ni}, x^*)\| \rightarrow 0$ (as $i \rightarrow \infty$)
where k is the normal constant of P.
Hence; $Tx_{ni} \rightarrow Tx^*$, $(i \rightarrow \infty)$
Similarly, $T^2x_{ni} \rightarrow T^2x^*$, $(i \rightarrow \infty)$
So, $D(Tx_{ni}, x_{ni}) \rightarrow D(Tx^*, x^*)$, $(i \rightarrow \infty)$

and $D(T^2x_{ni}, Tx_{ni}) \to D(T^2x^*, Tx^*)$, $(i\to\infty)$

It is obvious that: $D(Tx_{ni}, x_{ni}) = d_{ni} \to d^* = D(Tx^*, x^*)$, $(i\to\infty)$

Now, we shall prove that $Tx^*=x^*$. If $Tx^*\neq x^*$, then $d^*\neq 0$

We have: $d^* = D(Tx^*, x^*) > D(T^2x^*, Tx^*) = \lim_{i\to\infty} D(T^2x_{ni}, Tx_{ni}) = \lim_{i\to\infty} d_{n+1}$

$= d^*$

We have a contradiction, so $Tx^* = x^*$. That is x^* is a fixed point of T.The uniqueness of fixed point is obvious.

The following theorem improves **(Theorem 4.1.4)** by omitting the normality assumption of the cone P.

Theorem 4.1.6: [2] Let (X , D) be a complete cone metric space and the mapping $T: X \to X$ satisfies the contractive condition: for all $x, y \in X$ $D(Tx, Ty) \leq KD(x, y)$, where $K\in [0,1)$ is a constant. Then: T has a unique fixed point in X and for each $x \in X$, the iterative sequence $\{T^n x\}_{n\geq 1}$ converges to the fixed point.

Proof: For each $x_\circ \in X$ and $n \geq 1$

set $x_1 = Tx_\circ$ and $x_{n+1} = T^{n+1}x_\circ$, then:

$D(x_{n+1}, x_n) = D(Tx_n, Tx_{n-1}) \leq K\, D(x_n, x_{n-1})$

$$\leq K^2 D(x_{n-1}, x_{n-2}) \leq \ldots \leq K^n D(x_1, x_\circ)$$

So for $n > m$,

$D(x_n, x_m) \leq D(x_n, x_{n-1}) + D(x_{n-1}, x_{n-2}) + \ldots + D(x_{m+1}, x_m)$

$$\leq (K^{n-1} + K^{n-2} + \ldots + K^m)\, D(x_1, x_\circ) \leq \frac{K^m}{1-K} D(x_1, x_0)$$

Now, let $c \gg 0$ be given, choose $\delta > 0$ such that $c+N_\delta(0) \subseteq P$, where $N_\delta(0) = \{y \in E : \|y\| < \delta\}$.

Also, choose a natural number N_1 such that:

$\frac{K^m}{1-K}D(x_1,x_0) \in N_\delta(0), \ \forall\ m \geq N_1$. Then; $\frac{K^m}{1-K}D(x_1,x_0) \ll c, \ \forall\ m \geq N_1$

Thus, $D(x_n , x_m) \leq \frac{K^m}{1-K}D(x_1,x_0) \ll c \ , \ \forall n > m$.

Therefore, $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in (X , D), since (X , D) is a complete cone metric space, there is $x^* \in X$ such that $x_n \to x^*$.

Choose a natural number N_2 such that $D(x_n , x^*) \ll \frac{c}{2}$ for all $n \geq N_2$

Hence, $D(Tx^*, x^*) \leq D(Tx_n , Tx^*)+D(Tx_n , x^*) \leq K\ D(x_n , x^*)+D(x_{n+1} , x^*) \leq D(x_n , x^*)+D(x_{n+1} , x^*) \ll \frac{c}{2}+\frac{c}{2} = c \ , \ \forall n \geq N_2$ .Thus, $D(Tx^*, x^*) \ll \frac{c}{m}$, $\forall m \geq 1$. So $\frac{c}{m} - D(Tx^*, x^*) \in P, \forall\ m \geq 1$. Since $\frac{c}{m} \to 0$ (as $m \to \infty$) and P is closed, so $-D(Tx^*, x^*) \in P$. But $D(Tx^*, x^*) \in P$, therefore $D(Tx^*, x^*) = 0$ and so $Tx^* = x^*$.

Theorem 4.1.7 :[17] Let (X , D) be a complete cone metric space. For $c \in E$ with $c > 0$ and $x_\circ \in X$, set : $B(x_\circ , c)=\{x \in X : D(x_\circ , x) \leq c\}$**.** Suppose the mapping $T : X \to X$ satisfies the contractive condition : $D(Tx , Ty) \leq K\ D(x , y)$, for all $x ,y \in B(x_\circ , c)$ where $K \in [0 ,1)$ is a constant. Then T has a unique fixed point in $B(x_\circ , c)$ if and only if there exists $y \in B(x_\circ , c)$ such that: $D(Ty , y) \leq (1-K)(c-D(x_\circ , y))$.

The next theorem improves **Theorem 4.1.5** by omitting the regularity assumption and considering the weaker condition of normality on the cone P.

Theorem 4.1.8: [17] Let (X , D) be a sequentially compact cone metric space and P be a normal cone. Suppose the mapping $T: X \to X$ satisfies the contractive condition: $D(Tx ,Ty) < D(x , y)$, for all $x ,y \in X$ and $x \neq y$. Then T has a unique fixed point in X.

2. On generalization possibilities

In this section we obtain some examples in cone metric spaces in which some properties are incorrect but hold in ordinary case (metric space) and conversely. First example states that comparison test does not hold in cone metric spaces, the second example is for normal cones where we can find two members of the cone that f ≤ g but ‖ f ‖ ≥ ‖ g ‖. The last example is a contractive mapping on a cone metric space but not contractive in the Euclidean metric space. All this will convey the possibility of the idea of generalization.

As one notices, these examples appear elsewhere, but none of their authors made the remark that with these at hand, cone metric spaces somehow should generalize metric spaces.

Example 4.2.1: [13]Let E = $C_R^1[0,1]$ with norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and P={x ∈ E : x(t) ≥ 0} which is not a normal cone. For all n ≥ 1 and t ∈ [0 , 1] put $x_n(t) = \frac{t^{(n-1)^2}}{(n-1)^2+1} - \frac{t^{n^2}}{n^2+1}$ and $y_n(t) = \frac{2}{n^2}$ so $0 \le x_n \le y_n$ and $s_n(t) = \sum_{k=1}^{n} x_k(t) = 1 - \frac{t^{n^2}}{n^2+1}$.Therefore; $\|s_n - s_m\| = \|s_n - s_m\|_\infty + \|(s_n - s_m)'\|_\infty$ = $\left\| \frac{t^{m^2}}{m^2+1} - \frac{t^{n^2}}{n^2+1} \right\|_\infty + \left\| \frac{m^2 t^{m^2-1}}{m^2+1} - \frac{n^2 t^{n^2-1}}{n^2+1} \right\|_\infty = \frac{1}{m^2+1} + \frac{m^2}{m^2+1} =$ 1 , for all n, m. So $\{S_n\}$ is not a Cauchy sequence. Thus, $\sum_{k=1}^{\infty} x_k(t)$ is divergent, but $\sum_{k=1}^{\infty} y_k(t) = \sum_{k=1}^{\infty} \frac{2}{k^2}$ is convergent. This means that the comparison test does not hold for series.

Example 4.2.2: [13] Let E be a real vector space,

$E = \{ax + b \,;\, a , b \in \mathbb{R} , x \in [\frac{1}{2},1]\}$ with supremum norm and

$P = \{ax + b;\ a \leq 0,\ b \geq 0\}$. So P is normal cone in E with constant $k > 1$. Define $f(x) = -2x+10$, $g(x) = -6x+11$, $f, g \in P$, then $f \leq g$ since $g(x)-f(x) = -4x+1 \in P$. But, $\|f\| = f(\frac{1}{2}) = 9$ and $\|g\| = g(\frac{1}{2}) = 8$. Therefore, $f \leq g$ but $\|f\| \geq \|g\|$.

These examples lead us to find other examples or properties which may be held in ordinary spaces but don't hold in cone metric spaces.

Example 4.2.3: [1] Let $E = R^2$, and $P = \{(x, y) \in R^2 : x, y \geq 0\}$ a normal cone in E. Let $X = \{(x, 0) \in R^2 : 0 \leq x \leq 1\} \cup \{(0 , x) \in R^2 : 0 \leq x \leq 1\}$. The mapping $D: X \times X \to E$ is defined by:

$D((x , 0),(y , 0)) = (\frac{4}{3}|x-y| , |x-y|)$

$D((0 , x),(0 , y)) = (|x-y| , \frac{2}{3}|x-y|)$

$D((x , 0),(0 , y)) = D((0 , y),(x , 0)) = (\frac{4}{3}x+y ,\ x+\frac{2}{3}y)$

Then (X ,D) is a complete cone metric space. Let $T: X \to X$ defined by: $T((x , 0)) = (0 , x)$ and $T((0 , x)) = (\frac{1}{2}x , 0)$, then T satisfies the contractive condition: $D(T(x_1 , x_2), T(y_1 , y_2)) \leq KD((x_1 , x_2), (y_1 , y_2))$ for all $(x_1 , x_2),(y_1 , y_2) \in X$, with constant $K = \frac{3}{4}$. It is obvious that T has a unique fixed point $(0, 0) \in X$. On the other hand, we see that T is not a contractive mapping in the Euclidean metric R^2 on X.

Chapter Five

Measure theory in cone metric spaces

It is impossible to construct a function (measure m) operating on subsets of ℝ and sending them to the extended non-negative real numbers such that the followings all hold simultaneously:

(i)Domain of m = power set of ℝ.

(ii)m(I) = length of I; where I is any interval.

(iii)If $\{A_i\}$ is a pairwise disjoint sequence of sets in ℝ, then: $m(\bigcup_i A_i) = \sum_i m(A_i)$.

(iv) m has a translation invariant property;

i.e. $m(A) = m(A + x)$, ∀ A ⊆ ℝ and ∀ x ∈ ℝ.

It is not, yet, proven whether or not that m of this sort would satisfy the first three conditions. Even more we can say, specifically, it is known that if ℝ is equivalent (up to bijection) to each of its uncountable subsets, then no measure m can satisfy the first three conditions simultaneously. So we have to sacrifice at least one of the four given conditions.

For instance, relaxing the first condition or weakening it to only the requirement that (Domain of m ⊊ power set of ℝ) would make a good choice, though some sets of real numbers would not be measurable (i.e. have no images under m) and this is the choice we will take.

Now, let us rewrite the conditions after that brief introduction:

(i) $m(A) \in [0, \infty]$, ∀ A ⊆ ℝ.

(ii) m(I) = length of I; where I is any interval.

(iii) If $\{A_i\}$ is a pairwise disjoint sequence of sets in $\mathbb{R}$, then: $m(\bigcup_i A_i) = \sum_i m(A_i)$.

(iv) m has a translation invariant property;

i.e. $m(A) = m(A + x)$, $\forall A \subseteq \mathbb{R}$ and $\forall x \in \mathbb{R}$.

In this chapter we shall review the theory of Lebesgue measure, Lebesgue integral and Lebesgue integrable functions. Finally, we suggest some definitions for new concepts of measure theory in the sense of cone metric space, in order to compare some of the basics of Lebesgue measure theory.

Definition 5.1.1: [23]

a) A collection m of subsets of a set X is said to be a σ- algebra of X if it has the following properties:

(i) $X \in m$.

(ii) If $A \in m$, then $A^c \in m$, where A^c is the complement of A relative to X.

(iii) If $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in m$ for n = 1 , 2 , 3 , … , then $A \in m$.

b) If m is σ-algebra in X, then X is called a measurable space and the members of m are called the measurable sets in X.

c) If X is a measurable space, Y is a topological space, and f is a mapping of X into Y then f is said to be measurable provided that $f^{-1}(v)$ is a measurable set in X for every open set v in Y.

Definition 5.1.2: [23] For $E \subset X$, let $\mathcal{X}_E$ denote the characteristic function of E

i.e. $\mathcal{X}_E(t)=\begin{cases} 1, t \in E \\ 0, t \notin E \end{cases}$.

$\mathcal{X}_E(t)$ is measurable if and only if E is measurable.

Definition 5.1.3: [26]

a) A function $S: T \to \mathbb{R}$ is said to be simple if its range contains only finitely many points $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$.

b) If $S^{-1}(\alpha_i)$ is measurable for i = 1 , 2 , 3 , … , n then such a function can be written as: $S = \sum_{i=1}^{n} x_i \cdot \mathcal{X}_{A_i}$, where $A_i = S^{-1}(x_i)$, then we define:

$$\int_E s \, d\mu = \sum_{i=1}^{n} \alpha_i \cdot \mu(A_i \cap E) .$$

c)If f is a non-negative measurable function on E, then define:

$$\int_E f \, d\mu = \sup\{ \int_E s \, d\mu \; ; 0 \le s \le f , \text{ s is simple and measurable on E}\}$$

Definition 5.1.4: [23]

a) A measure μ is a function, defined on a σ- algebra m, whose range is in $[0, \infty]$ and which is countably additive. This means that if $\{A_n\}$ is a disjoint countable collection of members of m, then: $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.

b) A measure space is a measurable space which has a measure defined on the σ- algebra of its measurable sets.

c) A property which is true except for a set of measure zero is said to hold almost everywhere (a.e).

Remark5.1.5: [23] The following propositions are immediate consequences of the definitions. Functions and sets are assumed to be measurable on a measure space E;

a)If $0 \le f \le g$ then $\int_E f \le \int_E g$.

b) If $A \subset B$ and $f > 0$, then $\int_A f\, d\mu \le \int_B f\, d\mu$. (Monotonicity)

c)If c is a constant, then $\int_E c\, d\mu = c\, \mu(E)$.

d) If $E = E_1 \cup E_2$ where E_1 and E_2 are disjoint then: $\int_E f\, d\mu = \int_{E_1} f\, d\mu + \int_{E_2} f\, d\mu$.

e) If $f \ge 0$ and $\int_E f\, d\mu = 0$, then $f = 0$ a.e on E.

Definition5.1.6: [23] " The Lebesgue class $\mathcal{L}^1$ "

a) Let f be measurable on $\mathbb{R}$, then write: (i) $f = f^+ - f^-$

(ii) $|f| = f^+ + f^-$

where $f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$

b)Let $\mathcal{L}^1$ denote all measurable functions on $\mathbb{R}$ such that: $\int_{\mathbb{R}} |f| < \infty$

c)For $f \in \mathcal{L}^1$, write: (i) $\int_{\mathbb{R}} f = \int_{\mathbb{R}} f^+ - \int_{\mathbb{R}} f^-$

(ii) $\int_{\mathbb{R}} |f| = \int_{\mathbb{R}} f^+ + \int_{\mathbb{R}} f^-$

Note: Members of the class $\mathcal{L}^1$ are called Lebesgue integrable functions.

Now, let us mimic these terms from the beginning by replacing $\mathbb{R}$ by the Banach space in different places in the previous and see what will happen.

Definition 5.1.7: Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, where: Ω: set, $\mathcal{F}$: the set of all measurable sets in Ω and μ: measure. Let E be a real Banach space and P a cone in E. Fix a non-zero element of P and call it 1. For any subset A of Ω, we define:

a) The indicator function I_A , (instead of the characteristic function $\mathcal{X}_E$), by $I_A : \Omega \to E$; such that: $I_A(\omega) = \begin{cases} 1, \omega \in A \\ 0, \omega \notin A \end{cases}$

b) A simple function s on ω is one which takes the form : s : Ω → E such that: $s = \sum_{k=1}^{n} \alpha_k . I_{A_k}(\omega)$, k = 1 , 2 , … , n , Where $A_k \in \mathcal{F}$, $\alpha_k \in \mathbb{R}$.

c) For a non-negative simple function s ($\alpha_k \geq 0$, ∀ k) we define: $\int_{\Omega} s \, d\mu = \sum_{k=1}^{n} \alpha_k . \mu(A_k)$.

Definition 5.1.8: Suppose f ≥ 0 is a measurable function such that f: Ω → E, let $\mathcal{S}_f$ = {s: s is simple mble function s: Ω → E with s ≥ 0 and s(ω) ≤ f(ω), ∀ ω ∈ Ω}, then we can define:

$\int_{\Omega} f \, d\mu = \sup\{\int_{\Omega} s \, d\mu : s \in \mathcal{S}_f\}$.

Note: To ensure the well-definition of the last integral, we assume P is strongly minihedral, f bounded on Ω (i.e.: ∃ z ∈ P such that: f(ω) ≤ z, ∀ ω ∈ Ω) and that μ is a finite measure.

Now, for any measurable function f on Ω, assume P is a strongly minihedral cone in E, then:

(i)$f^+(\omega) = \sup\{f(\omega), 0\}$

(ii) $f^-(\omega) = \sup\{-f(\omega), 0\}$. So that $f(\omega) = f^+(\omega) - f^-(\omega)$

Hence, we can define: $\int_{\Omega} f \, d\mu = \int_{\Omega} f^+ \, d\mu - \int_{\Omega} f^- \, d\mu$

Finally, we also present a famous example herein.

Example 5.1.9: [13] Let (Ω, S, μ) be a finite measure space. S: countably generated, assume that $E = L^p(\Omega)$, $1 \leq p < \infty$ and $P = \{f \in E: f(\omega) \geq 0 \mu$ a.e. on $\Omega\}$ then this cone P is normal, regular, minihedral and strongly minihedral and it is not solid ($P^{\circ} = \emptyset$).

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جامعة النجاح الوطنية
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فضاءات القياسات المخروطية المترية

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قدمت هذه الاطروحة استكمالا لمتطلبات الحصول على درجة الماجستير في الرياضيات بكلية الدراسات العليا في جامعة النجاح الوطنية في نابلس – فلسطين
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الملخص

مؤخرا في عام 2007 م تم طرح موضوع الفضاءات المخروط المترية ونتج عن ذلك العديد من الابحاث والاوراق العلمية، من هذه الاوراق ما هو متعلق بنظريات النقطة الثابتة (fixed point theorems) ومنها ما هو متعلق بالبناء الرياضي لهذه الفضاءات المخروطية المترية.

في الواقع كانت فكرة انتاج الفضاءات المخروطية المترية تتخلص في استبدال المجال المقابل لاقتران الفضاءات المترية العادية وهو مجموعة الاعداد الحقيقية ($\mathbb{R}$) بفضاء باناخ (Banach space) ونتج عندئذ فضاء القياس المخروط المتري. وهنا نشأ سؤال مهم وهو محور رسالتنا: هل الفضاء المخروطي المتري تعميم للفضاء المتري ام انهما متكافئان ؟

تمت الاجابة عن هذا السؤال في العديد من الاوراق العلمية وبطرق متعددة، ونحن ايضا اسهمنا في الاجابة عن السؤال بان الفضاءان متكافئان وذلك بورقة بحثية تحت عنوان

(Metrizability of Cone Metric Spaces Via Renorming the Banach Spaces)

وجاءت الموافقة لنشر هذه الورقة مؤخرا في المجلة العلمية الالمانية (JNAA) وعليه قمنا ببناء موضوع الرسالة المكونة من خمس فصول تحدثنا فيها عن بعض مواضيع الرياضيات البحتة ولكن بصورة جديدة وهي صورة (cone metric spaces) بدلا من (metric spaces) .